Neo-Riemannian Operations, Parsimonious Trichords, and Their "Tonnetz" Representations
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NEO-RIEMANNIAN OPERATIONS, PARSIMONIOUS TRICHORDS, AND THEIR *TONNETZ* REPRESENTATIONS

Richard Cohn

1. The Over-Determined Triad

In work published in the 1980s, David Lewin proposed to model relations between triads[1] using operations adapted from the writings of the turn-of-the-century theorist Hugo Riemann.[2] Subsequent work along neo-Riemannian lines has focused on three operations that maximize pitch-class intersection between pairs of distinct triads: P (for Parallel), which relates triads that share a common fifth; L (for Leading-tone exchange), which relates triads that share a common minor third; and R (for Relative), which relates triads that share a common major third.[3] Figure 1 illustrates the three operations, which I shall refer to collectively as the **PLR family**, as they act on a C minor triad, mapping it to three different major triads. (Throughout this paper, + and - are used to denote major and minor triads, respectively.) Singly applied, each PLR-family operation inverts a triad (major ⟷ minor). Doubly applied, each PLR-family operation maps a triad to its identity; i.e., each is an involution.

A striking feature of PLR-family operations is their parsimonious voice-leading.[4] To a degree, parsimony is inherent to the PLR family, whose defining feature is double common-tone retention. What is not inherent is the incremental motion of the third voice, which proceeds by

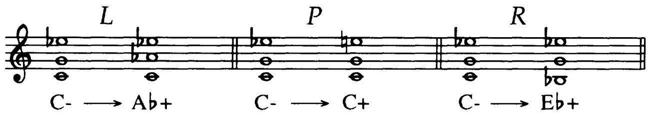


Figure 1: The PLR-Family of C-Minor

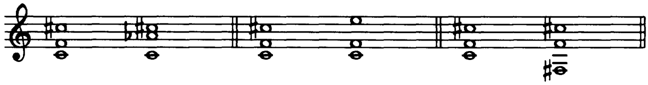


Figure 2: The PLR-Family of {0, 1, 5}

semitone in the case of P and L, and by whole tone in the case of R. This feature is not without significance to the development of a musical culture where conjunct voice-leading in general, and semitonal voice-leading in particular, are enduring norms through an impressive range of chronological eras and musical styles.[5] The parsimony of PLR-family voice-leading is so engrained in the procedural knowledge of a musician trained in the European classical tradition that it hardly seems to warrant notice, much less scrutiny. It is scrutinized here with the aim of demonstrating that, from a certain point of view, the feature is fortuitous.

In order to cultivate this point of view, imagine a musical culture where set-class 3-4 (015) was the privileged harmony to the extent that the triad prevails in European music c. 1500–1900. For any member of set-class 3-4, there are three other members with which it shares two common pcs. Figure 2 leads the prime form of 3-4 to its three double-common-tone related peers. Note the magnitude of the moving voice: in two cases by minor third, in the third case by tritone. Such a culture would be incapable of achieving the degree of voice-leading parsimony characteristic of triadic music, particularly as it developed in the nineteenth century. It can be easily verified, and will be demonstrated shortly, that replication of the exercise using any other mod-12 trichord-class yields similarly unparsimonious results.[6]

It may come as no surprise that, among trichord-classes, consonant triads are special. Their unique acoustic properties are well established, and indeed are fundamental to standard approaches to triadic music. The potential of consonant triads to engage in parsimonious voice-leading, however, is unrelated to those acoustic properties. This potential is, rather, a function of their group-theoretic properties as equally tempered entities modulo-12.

To demonstrate this claim, the definition of PLR-family operations is now generalized, initially to the prime forms of set-classes defined by T_n/T_nI equivalence, subsequently to all trichords. Although our primary attention will be focused on trichords in the usual chromatic system of twelve pitch-classes, the definition is open to application to other systems as well, for reasons that will emerge as the exposition unfolds.

DEF. (1). **c** is a positive integer representing the cardinality of a chromatic system.

DEF. (2). **Q** is a mod-c trichord {0, x, x + y} such that $0 < x \leq y \leq c - (x + y)$.

The condition insures that Q is the prime form of its trichord-class.

DEF. (3) I_u^v is the inversion that maps pitch-classes v and u to each other.[7]

The three PLR-family operations can now be defined on a prime-form trichord Q = {0, x, x + y} as follows:

DEF. (4a) P = I^0_{x+y}	DEF. (4b) L = I^0_x	DEF. (4c) R = I^x_{x+y}

Figure 3a (p. 4) demonstrates the mapping of abstract pitch-classes when P, L, and R act on the abstract trichordal prime form Q. Figures 3b and 3c realize Q as the two trichords explored in Figures 1 and 2. Each operation swaps two of the pitch-classes in Q. The remaining pc is mapped outside of Q, and this mapping is perceived as the "moving voice." We now define a set of variables, $\mathscr{p}$, $\mathscr{l}$, and $\mathscr{r}$, which express the magnitude of those external mappings as mod-c transpositional values.

$x \xrightarrow{P} y$, hence: DEF. (5a) $\mathscr{p}$ = y - x	$x + y \xrightarrow{L} -y$, hence: DEF. (5b) $\mathscr{l}$ = -2y - x	$0 \xrightarrow{R} 2x + y$, hence: DEF. (5c) $\mathscr{r}$ = 2x + y

Observe that $\mathscr{p} + \mathscr{l} + \mathscr{r} = 0$, since (y - x) + (-2y - x) + (2x + y) = (2x - 2x) + (2y - 2y) = 0.

The values of $\mathscr{p}$, $\mathscr{l}$, and $\mathscr{r}$ are now linked to the structure of trichord Q. First, following Bacon (1917), Chrisman (1971), and others since, we define a step-interval series as follows:

DEF (6). The **step-interval series** for a normal-order trichord {i,j,k} is the ordered set <j-i, k-j, i-k>, modulo c.

Via Def. (2), Q = {0, x, x+ y} is in prime form, which presupposes normal order. Thus the step-interval series of Q is <x, y, - (x + y)>.

$P(Q) = I^{0}_{x+y}(Q)$	$L(Q) = I^{0}_{x}(Q)$	$R(Q) = I^{x}_{x+y}(Q)$
$x + y \rightarrow 0$	$x + y \rightarrow -y$	$x + y \rightarrow x$
$x \rightarrow y$	$x \rightarrow 0$	$x \rightarrow x + y$
$0 \rightarrow x + y$	$0 \rightarrow x$	$0 \rightarrow 2x + y$

Figure 3: PLR-Family Mappings
(a) for Q = {0, x, x + y}

$P(Q) = I^{0}_{7}(Q)$	$L(Q) = I^{0}_{3}(Q)$	$R(Q) = I^{3}_{7}(Q)$
$7 \rightarrow 0$	$7 \rightarrow 8$	$7 \rightarrow 3$
$3 \rightarrow 4$	$3 \rightarrow 0$	$3 \rightarrow 7$
$0 \rightarrow 7$	$0 \rightarrow 3$	$0 \rightarrow 10$

Figure 3: PLR-Family Mappings
(b) for C-Minor; x = 3, y = 4, Q = {0, 3, 7}

$P(Q) = I^{0}_{5}(Q)$	$L(Q) = I^{0}_{1}(Q)$	$R(Q) = I^{1}_{5}(Q)$
$5 \rightarrow 0$	$5 \rightarrow 8$	$5 \rightarrow 1$
$1 \rightarrow 4$	$1 \rightarrow 0$	$1 \rightarrow 5$
$0 \rightarrow 5$	$0 \rightarrow 1$	$0 \rightarrow 6$

Figure 3: PLR-Family Mappings
(c) x = 1, y = 4, Q = {0, 1, 5}

The following theorem states that p, l, and r are each equivalent to the differences between a distinct pair of step-intervals.

THEOREM 1. For a prime-form trichord Q = {0, x, x + y} with step-intervals <x, y, - (x + y)>,

1.1) p is the difference between the first and second step interval, y - x. Proof: y - x = p via Def. (5a).

1.2) l is the difference between the second and third step interval, - (x+y) - y. Proof: - (x+y) - y = - 2y - x = l via Def. (5b).

1.3) r is the difference between the third and first step interval, x - (-(x+y)). Proof: x - (- (x+y)) = 2x + y = r via Def. (5c).

Theorem 1 facilitates calculation of the values of p, l, and r for any trichordal prime form in a chromatic system of any size. The results of these calculations for the mod-12 trichordal prime forms are given in Figure 4 (p. 6). Conjunct intervals are enclosed in boxes. The figure demonstrates what was asserted above: that 037 is unique among trichordal prime forms (modulo 12) in its preservation of conjunct melodic intervals when any of the three PLR-family operations is executed.[8]

The generalization from prime form to other class members is easily carried out. Calculate the T_n or T_nI of the trichord in relation to the prime form of its class, reassign 0 to the pitch-class that had been formerly assigned the integer n, and reassign the integers in ascent or descent from the new pitch-class 0, depending on whether the trichord is T_n or T_nI-related to the prime form. Then apply the operations as in definition (4). If Tn-related to the prime form, the values of *p*, *l*, and *r* are the same as when the operation acts on the prime form; if TnI-related, the values are inverted. (This follows from the involutional nature of PLR-family operations.) In either case, the magnitudes are invariant. Consequently, the unique characteristic noted above for trichordal prime form {037} generalizes to all members of its set-class.

To summarize our findings so far: (1) Among mod-12 trichords, the consonant triad alone is susceptible to parsimonious voice-leading under the three PLR-family operations; (2) This circumstance is a function of the trichord's step-interval sizes, which are an aspect of its internal structure; (3) the optimal voice-leading properties of triads therefore stand in incidental relation to their optimal acoustic properties.

In a word: the triad is over-determined.

The fortuitous relation of the consonant triad's voice-leading parsimony to its acoustic generability is as profound to the development of the European musical tradition as other sorts of over-determination that were first brought to light at Princeton in the 1960s (Babbitt 1965, Gamer 1967, Boretz 1970): of the chromatic division of the octave into twelve parts, 12 being at once the smallest abundant integer and the smallest integer n such that 3^n approximates some power of 2; of the proximity of the perfect fifth's $\frac{2}{3}$ geometric division of the octave (the source of its acoustic power) to its $\frac{7}{12}$ arithmetic division of the octave (a fraction whose irreducibility, rare for its divisor, is necessary for the deep-scale property of diatonic collections, a circumstance which in turn leads to the graded common-tone distribution of the set of diatonic collections under transposition, and hence to the system of key signatures). Equally remarkable is the extent to which the triad's acoustic properties have masked recognition of its group-theoretic potential.[9] Our sensibilities, born of incessant exposure to a musical tradition that habitually implements the acoustic properties of triads, as well as to a music-theoretic tradition that habitually models this habitual implementation, have been trained to resist by default any effort to regard the triad as anything other than acoustic *in essence*. Like the stock figure of the Cold War spy

prime form (0, x, x + y)	step-intervals <x, y, - (x + y)>	*p* y -x	*l* -2y - x	*r* 2x + y
{0,1,2}	<1,1,10>	0	9	3
{0,1,3}	<1,2,9>	$\boxed{1}$	7	4
{0,1,4}	<1,3,8>	$\boxed{2}$	5	5
{0,1,5}	<1,4,7>	3	3	6
{0,1,6}	<1,5,6>	4	$\boxed{1}$	7
{0,2,4}	<2,2,8>	0	6	6
{0,2,5}	<2,3,7>	$\boxed{1}$	4	7
{0,2,6}	<2,4,6>	$\boxed{2}$	$\boxed{2}$	8
{0,2,7}	<2,5,5>	3	0	9
{0,3,6}	<3,3,6>	0	3	9
{0,3,7}	<3,4,5>	$\boxed{1}$	$\boxed{1}$	$\boxed{10}$
{0,4,8}	<4,4,4>	0	0	0

Figure 4: Values of *p*, *l*, and *r* for trichordal prime forms

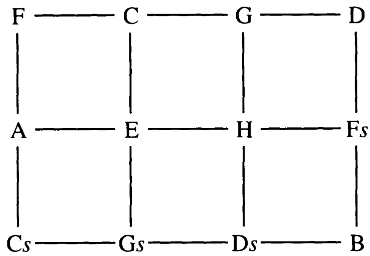


Figure 5: Two *Tonnetze*
(a) from Euler 1926 (1739)

thriller, the dazzling beauty of the triad has blinded us to its substantial intellectual resources.

2.1. The Over-Determined *Tonnetz*

The two-dimensional matrix known as the table of consonant (or tonal) relations (*Verwandtschaftsverhältnistabelle*) or harmonic network (*Tonnetz, Tongewebe*) has long presented music theorists with a useful graphic instrument for representing triadic progressions. Although the matrix originated in response to the acoustic properties of triads, it responds in equal measure to their group-theoretic properties. The over-determination of the triad is thus encoded in the over-determination of the *Tonnetz.*

The *Tonnetz* was initially conceived to reconcile the first two distinct (non-complementary) sub-octave intervals generated from a resonating body.[10] Leonhard Euler (1926 [1739]) situated justly tuned versions of the twelve pitch classes on a bounded 4×3 matrix whose axes are generated by acoustically pure fifths (3:2) and major thirds (5:4) (Figure 5a).[11] Arthur von Oettingen (1866, 15) inverted Euler's matrix about the horizontal axis, and projected it onto an infinite plane, as shown in Figure 5b (p. 8). (The slashes represent syntonic comma adjustments, and result from Oettingen's sensitivity to the just-intonational distinctions masked by notational and letter-name equivalence.) This version of the pitch-class table was appropriated by Riemann, became widely disseminated through his writings, and has been passed down by generations of German harmonic theorists leading all the way up to the present day (see Imig 1970, Harrison 1994).

m 2	$\overline{\overline{\text{c}}}$	$\overline{\overline{\text{g}}}$	$\overline{\overline{\text{d}}}$	$\overline{\overline{\text{a}}}$	$\overline{\overline{\text{e}}}$	$\overline{\overline{\text{h}}}$	$\overline{\overline{\text{fis}}}$	$\overline{\overline{\text{cis}}}$	$\overline{\overline{\text{gis}}}$	$\overline{\overline{\text{dis}}}$	$\overline{\overline{\text{ais}}}$	$\overline{\overline{\text{eis}}}$	$\overline{\overline{\text{his}}}$	$\overline{\overline{\text{fisis}}}$	$\overline{\overline{\text{cisis}}}$	$\overline{\overline{\text{gisis}}}$	$\overline{\overline{\text{disis}}}$
1	$\overline{\text{as}}$	$\overline{\text{es}}$	$\overline{\text{b}}$	$\overline{\text{f}}$	$\overline{\text{c}}$	$\overline{\text{g}}$	$\overline{\text{d}}$	$\overline{\text{a}}$	$\overline{\text{e}}$	$\overline{\text{h}}$	$\overline{\text{fis}}$	$\overline{\text{cis}}$	$\overline{\text{gis}}$	$\overline{\text{dis}}$	$\overline{\text{ais}}$	$\overline{\text{eis}}$	$\overline{\text{his}}$
0	fes	ces	ges	des	as	es	b	f	c	g	d	a	e	h	fis	cis	gis
– 1	$\underline{\text{deses}}$	$\underline{\text{asas}}$	$\underline{\text{eses}}$	$\underline{\text{bb}}$	$\underline{\text{fes}}$	$\underline{\text{ces}}$	$\underline{\text{ges}}$	$\underline{\text{des}}$	$\underline{\text{as}}$	$\underline{\text{es}}$	$\underline{\text{b}}$	$\underline{\text{f}}$	$\underline{\text{c}}$	$\underline{\text{g}}$	$\underline{\text{d}}$	$\underline{\text{a}}$	$\underline{\text{e}}$
– 2	$\underline{\underline{\text{bbb}}}$	$\underline{\underline{\text{feses}}}$	$\underline{\underline{\text{ceses}}}$	$\underline{\underline{\text{geses}}}$	$\underline{\underline{\text{deses}}}$	$\underline{\underline{\text{asas}}}$	$\underline{\underline{\text{eses}}}$	$\underline{\underline{\text{bb}}}$	$\underline{\underline{\text{fes}}}$	$\underline{\underline{\text{ces}}}$	$\underline{\underline{\text{ges}}}$	$\underline{\underline{\text{des}}}$	$\underline{\underline{\text{as}}}$	$\underline{\underline{\text{es}}}$	$\underline{\underline{\text{b}}}$	$\underline{\underline{\text{f}}}$	$\underline{\underline{\text{c}}}$

Figure 5: Two *Tonnetze*
(b) from Oettingen 1866

The position of major and minor triads on the matrix was first observed by Euler (1926 [1773], 585), who noted that they could be represented on his matrix by the conjunction of two perpendicular line-segments. Oettingen, less concerned than Euler about the acoustically residual status of the minor third, suggested adding a hypotenuse to close the structure to a right triangle (1866, 17). This move brings PLR-family relations to the forefront, since triads so related are represented by triangles that share an edge, and are thereby maximally proximate. One might infer from this circumstance that PLR-family operations would be the vehicle of choice for navigating triadic progressions on the *Tonnetz*, but this has not been the case historically. The development of Oettingen's table as a "gameboard" for mapping progressions among triads was instead guided by convictions about the acoustic, tonally centric status of consonant triads and their relations to each other. This resulted in the subordination of PLR-family relations to the Tonic/Subdominant/Dominant (TSD) "functional" framework developed by Riemann in the 1890s, a framework that has continued to dominate Northern European harmonic theory ever since.[12]

The mapping of PLR-family operations independently of the TSD framework was first proposed by Lewin (1987, 175–180) and has been developed by Brian Hyer (1989, 1995), whose work demonstrates the heuristic value of charting PLR-family operations on the *Tonnetz* without necessary recourse to assumptions about tonal centers, TSD-functional relations, or the acoustic properties of triads. The emphasis on PLR-family operations in the work of Lewin and Hyer is apparently motivated empirically by the desire to model characteristically late-Romantic progressions in a manner that is faithful to the musical qualities that they are perceived to project. The focus on voice-leading parsimony cultivated in Section 1 above suggests a complementary deductive-rationalist motivation for liberating the triad, PLR-family operations, and their *Tonnetz* representation from their acoustic origins.

It is this motivation that directs the remaining program for this paper. In the material that follows immediately, a version of the *Tonnetz* of Oettingen and Riemann is situated within a genus and species of two-dimensional matrices. The defining characteristic of the genus is that pairs of trichords represented by adjacent triangles are related by PLR-family operations, as broadly defined in Section 1 above (cf. Def. (4)). The defining characteristic of the species is that pairs of trichords represented by adjacent triangles feature parsimonious voice-leading. Section 2.4 refines the conception of the Oettingen/Riemann matrix, and of the species that it represents, by acknowledging the toroidal geometry that underlies them when their contents and relations are interpreted in the context of equal temperament. The focus on the *Tonnetz* throughout Section 2 serves as a large structural upbeat to Section 3, the musical core of

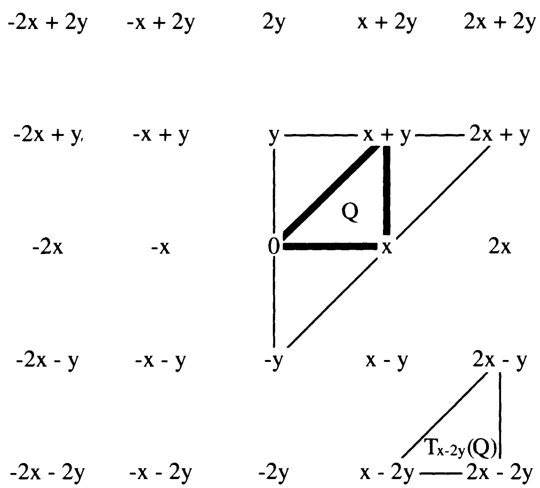


Figure 6: The Abstract *Tonnetz*

the paper, which uses PLR-family operations to navigate the *Tonnetz*, in its various versions at various degrees along the abstraction/specification continuum.

2.2. The Generic *Tonnetz*

Our investigation of the *Tonnetz* of harmonic theory initially situates it as a member of an infinite class of two-dimensional matrices whose generic form is presented in Figure 6. The primary axes of Figure 6 are generated by the generic intervals x and y, in the sense that each row increments from left to right by the value of x, and each column increments from bottom to top by the value of y. The figure should be interpreted as projecting its structure beyond its boundaries. The elements of Figure 6 represent real numbers as they increment to infinity, and should not be interpreted in the context of the closed modular systems that were the focus of our previous work. Figure 6 is neither more nor less than the Cartesian coordinate plane of analytic geometry.

In terms of Def. (2), the primary axes of Figure 6 are interpreted as the smallest two step-intervals of a prime form trichord Q = {0, x, x+y} with step-intervals <x, y, -(x+y)>. The remaining step-interval is the inverse of the sum of the two smaller step-intervals, and hence generates the diagonal which runs from northeast to southwest, in the sense that each such diagonal decrements, sloping southwestward, by x+y.

Figure 6 represents trichord Q as a darkly bordered triangle at its center. Each vertex of the triangle represents a pitch or pitch-class,[13] and each edge a dyadic subset, of Q. Any geometric translation of this triangle—that is, any triangle whose hypotenuse subtends northwest of the right angle—represents a pitch or pitch-class transposition of Q. (An example, labeled $T_{x-2y}(Q)$, is provided by the isolated triangle in the southeast corner of Figure 6.) Furthermore, any geometric inversion of the Q triangle about a secondary diagonal—that is, any triangle whose hypotenuse subtends southeast of the right angle—represents a pitch or pitch-class inversion of Q. Figure 6 indicates three such inverted triangles, all sharing an edge with the central triangle. These three triangles represent the PLR-family of Q (cf. Figure 3a). The unique edge that the central triangle shares with each of its adjacent triangles represents the unique dyad that trichord Q shares with each member of its PLR-family.

Figure 7 replicates the core of Figure 6 and adds three arrows, each labeled with one of the voice-leading variables from Def. (5). Each arrow represents the magnitude of the moving voice when Q is subject to a PLR-family operation:

- *p* labels $x \longrightarrow y$ when P takes {0, x, x + y} to {0, y, x + y} across their shared hypotenuse;
- *l* labels $x + y \longrightarrow -y$ when L takes {0, x, x + y} to {0, x, - y} across their shared horizontal edge;
- *r* labels $0 \longrightarrow 2x + y$ when R takes {0, x, x + y} to {2x + y, x, x + y} across their shared vertical edge.

When the same operations are enacted on other members of trichord-class Q, their geometric orientation on Figure 7 is invariant. In all cases, P-related triads share a hypotenuse, and *p* executes a pawn-capture along the main diagonal; L-related triads share a horizontal edge, and *l* executes a knight's move, a displacement by two rows and one column; R-related triads share a vertical edge, and *r* executes a knight's move, a displacement by two columns and one row. The magnitudes of *p*, *l*, and *r* are likewise invariant, although when the object of the mapping is a triad T_nI-related to the prime form, the direction of the arrow reverses, and the values of *p*, *l*, and *r* invert.

The unbounded grid inferable from Figure 6 is applicable to pitches and intervals in a variety of ways. If **x** and **y** are assigned to acoustically pure intervals (as in Euler, etc.), or to intervals in pitch-space, then the structure implicitly projects into an infinite plane. The realizations of the Figure 6 grid that will hold our focus are generated by equally tempered intervals in some modular system, where the modular congruence repre-

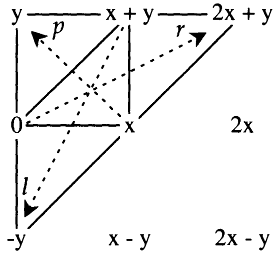


Figure 7: *Tonnetz* Representation of
PLR-Family Operations on Q = {0, x, x + y}

sents octave equivalence. In such interpretations, both x and y axes become cyclic rather than linear, and the plane inferred from Figure 6 therefore projects into itself as a torus.[14] These cyclic features will be studied in some detail in Section 2.4 below.

2.3. The Parsimonious *Tonnetz*

In Figure 7, PLR-family operations are only associated with voice-leading parsimony in the restricted sense that each operation preserves two common tones. The degree of parsimony associated with the third voice depends on the magnitudes of *p*, *l* and *r*, which in turn depend on the values of **x** and **y**, as yet unassigned. Furthermore, the interpretation of that magnitude as representing an interval-class in a modular pitch-class system depends on the imposition of a congruence, i.e. a specific value for **c**. The first section of this paper established that, where c = 12, the three PLR-family operations are parsimonious only when x = 3 and y = 4, i.e when the trichord-class is the consonant triad, in which case the generic *Tonnetz* is realized as an equal-tempered version of the Oettingen/Riemann *Tonnetz* of Figure 5b. Since this is the case of historical and analytical interest, it will soon be subject to detailed scrutiny. First, however, it will be instructive to consider a structure of intermediate abstraction; a "middleground" *Tonnetz* that "composes out" Figure 6 in a particular way, at the same time as it positions the properties of the Oettingen/Riemann *Tonnetz* in a general context.

What values of c, x, and y will lead to optimally parsimonious voice-leading when PLR-family operations are executed? Intuitively, the degree of parsimony is optimal when the magnitudes (i.e. absolute values) of the voice-leading intervals *p*, *l* and *r* are as small as possible, but greater than zero. (The last condition insures that voice-leading "motion" is perceptible as such; cf. note 6.) Ideal parsimony, then, would be achieved when $\mathscr{p} = \pm 1$, $\mathscr{l} = \pm 1$, and $\mathscr{r} = \pm 1$. But this combination is impossible. As observed in Section 1, $\mathscr{p} + \mathscr{l} + \mathscr{r} = 0$, and so each variable is the inverse of the sum of the other two. Consequently, $\mathscr{p}$, $\mathscr{l}$, and $\mathscr{r}$ must in this case have identical directions as well as magnitudes, and so $\mathscr{p} = \mathscr{l} = \mathscr{r}$. Via Defs. (5b) and (5c), if $\mathscr{l} = \mathscr{r}$ then $- 2y - x = 2x + y$, and so $3x = - 3y$, hence $x = - y$, which implies that $y - x$ is even. Thus $\mathscr{p}$ is even (via Def. (5a)), and so $\mathscr{p} \neq \pm 1$, contrary to what was stipulated.

The next recourse is to increment the magnitude of one of the voice-leading variables. In principle, any of the three variables can be incremented, but Q is in prime form (cf. Def. 2) only if $\mathscr{p} = 1$, $\mathscr{l} = 1$, $\mathscr{r} = -2$. These are exactly the values for the familiar case of the consonant triad modulo 12. Once we cease to assume a chromatic system of twelve pitch-classes, as we did in section 1, what other combinations lead to these values of $\mathscr{p}$, $\mathscr{l}$, and $\mathscr{r}$? This problem is easily solved using Theorem 1, which linked the voice-leading intervals to step-interval differences. If the first step interval is x, and $\mathscr{p} = 1$, then the second step interval is x + 1, via Theorem 1.1. Further, since $\mathscr{l} = 1$, then the third step interval is x + 2, via Theorem 1.2. The three step intervals of a parsimonious trichord, then, must form the ascending consecutive series <x, x + 1, x + 2>.[15]

The sum of these three step intervals is 3x + 3, distributed as 3(x + 1). Since c, the number of pitch-classes in the system, is the sum of the step-intervals, it follows that parsimonious trichords are only available if c is an integral multiple of 3. In each such system, there is a single step-interval series of the form <x, x + 1, x + 2> that represents a parsimonious trichord-class whose prime form is {0, x, 2x + 1}. The generic version of such a trichord will be represented using the variable Q′:

DEF. (7). Q′ = {0, x, 2x + 1}, modulo 3x + 3

Figure 8 (p. 14) presents a generic *Tonnetz* for parsimonious trichords, which will be referred to as the **parsimonious *Tonnetz*** for short. The axes of Figure 8 are generated by x and x + 1, the smallest step-intervals in Q′. A modular congruence of 3x + 3 is imposed on the figure, so that the first term of each expression is represented as a non-negative value. The trichordal prime-form Q′ = {0, x, 2x + 1} is portrayed along with its PLR family at the center of Figure 8 . The arrows depict the following:

- $\mathscr{p}$ labels the T_1 motion x ⟶ x + 1 when P takes {0, x, 2x + 1} to {0, x + 1, 2x + 1} across their shared hypotenuse;
- $\mathscr{l}$ labels the T_1 motion 2x + 1 ⟶ 2x + 2 when L takes {0, x, 2x + 1} to {0, x, 2x + 2} across their shared horizontal edge;

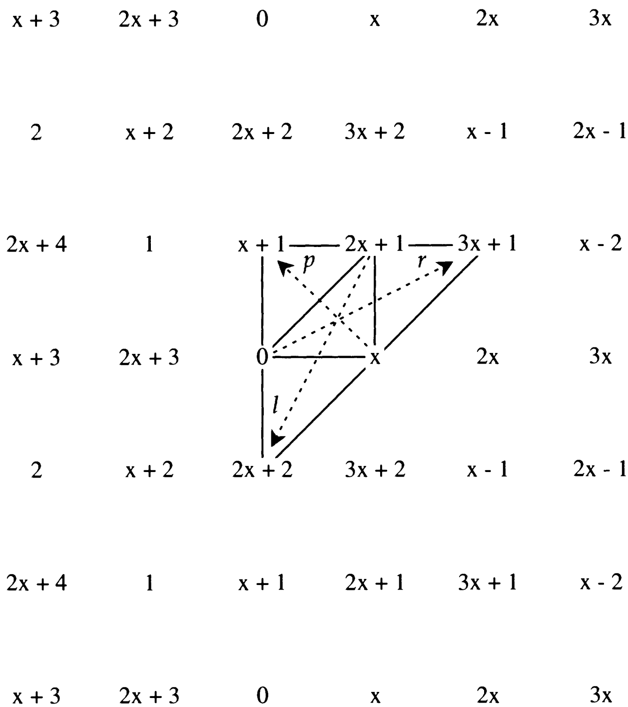


Figure 8: The Parsimonious *Tonnetz*

- *r* labels the T_{-2} motion $0 \longrightarrow 3x + 1 \equiv -2$ when R takes {0, x, 2x + 1} to {3x + 1, x, 2x + 1} across their shared vertical edge.

Figure 8 is a powerful representation of parsimonious trichordal motion. No matter what value is assigned to x, any local triangle with a southwest/northeast hypotenuse represents a parsimonious trichord. Conversely, all parsimonious trichords are representable through a realization of Figure 8. To investigate the scope of this power, we now explore three

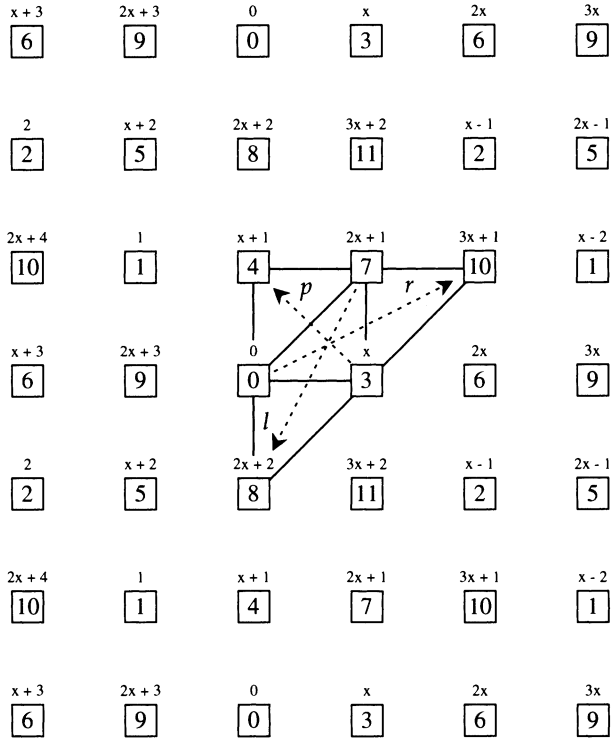


Figure 9: Three Realizations of the Parsimonious *Tonnetz.*
(a) x = 3, y = 4, c = 12, Q' = {0, 3, 7} modulo 12
(Oettingen / Riemann *Tonnetz*)

such realizations, for x = 3, 5, and 7 respectively. In each of the three matrices that comprise Figure 9, the abstract expressions of Figure 8 are retained along with their realizations so that derivations can be easily traced.

Figure 9a is a version of the Oettingen/Riemann *Tonnetz.* It partially rotates Figure 5b, replacing pitch-class names with integers. The series of major thirds is retained on the y axis, but the series of minor thirds is displaced from the northwest/southeast diagonal of Figure 5b to the x axis of Figure 9a, thereby shifting the series of perfect fifths from the x axis

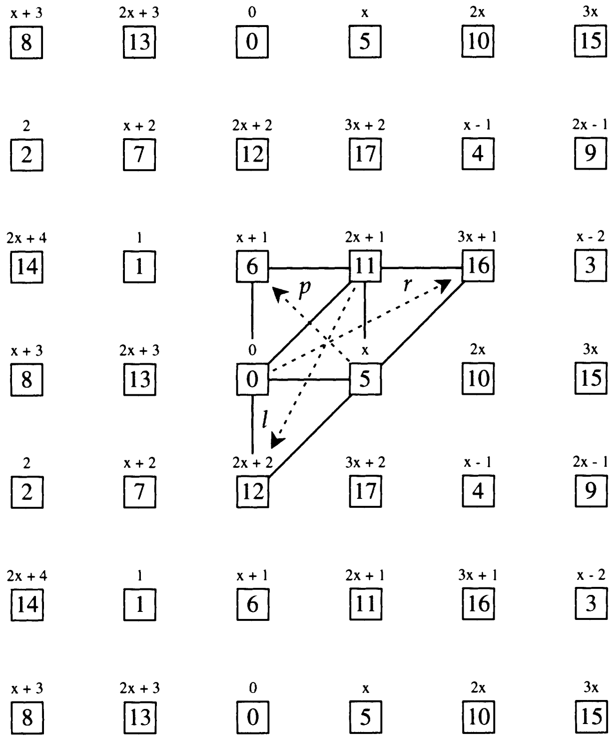


Figure 9: Three Realizations of the Parsimonious *Tonnetz.*
(b) x = 5, y = 6, c = 18, Q' = {0, 5,11} modulo 18

to the southwest/northeast diagonal. This particular version of the *Tonnetz* actually antedates Oettingen: it was first introduced by Carl Friedrich Weitzmann in 1853, although in a bounded form (as in Fig. 5a), and using staff-notated pitches rather than integers.[16] The triangular complex at the center of Figure 9a portrays the trichordal prime form, {0, 3, 7} = C minor, together with its PLR family. The arrows represent the following:

- *p* labels the T_1 semitonal motion $3 \longrightarrow 4 = E\flat \longrightarrow E$ when P takes {0, 3, 7} = C minor to {0, 4, 7} = C major across their shared hypotenuse;
- *l* labels the T_1 semitonal motion $7 \longrightarrow 8 = G \longrightarrow A\flat$ when L takes {0, 3, 7} = C minor to {0, 3, 8} = A♭ major across their shared horizontal edge;
- *r* labels the $T_{10} \equiv T_{-2}$ whole-step motion $0 \longrightarrow 10 = C \longrightarrow B\flat$ when R takes {0, 3, 7} = C minor to {10, 3, 7} = E♭ major across their shared vertical edge.

In Figure 9b, the parsimonious trichord is Q′ = {0, 5, 11} in a modulo 18 ("third-tone") system, with step-intervals <5, 6, 7>. The triangular complex at the center of Figure 9b portrays {0, 5, 11} together with the three trichords that comprise its PLR family. The arrows portray the following:

- *p* labels the T_1 motion $5 \longrightarrow 6$ when P takes {0, 5, 11} to {0, 6, 11} across their shared hypotenuse;
- *l* labels the T_1 motion $11 \longrightarrow 12$ when L takes {0, 5, 11} to {0, 5, 12} across their shared horizontal edge;
- *r* labels the $T_{16} \equiv T_{-2}$ motion $0 \longrightarrow 16$ when R takes {0, 5, 11} to {16, 5, 11} across their shared vertical edge.

In Figure 9c (p. 18), the parsimonious trichord is Q′ = {0, 7, 15} in a modulo 24 ("quarter-tone") system, with step-intervals <7, 8, 9>. The triangular complex at the center of Figure 9c portrays {0, 7, 15} together with the three trichords that form its PLR family. The arrows portray the following:

- *p* labels the T_1 motion $7 \longrightarrow 8$ when P takes {0, 7, 15} to {0, 8, 15} across their shared hypotenuse;
- *l* labels the T_1 motion $15 \longrightarrow 16$ when L takes {0, 7, 15} to {0, 7, 16} across their shared horizontal edge;
- *r* labels the $T_{22} \equiv T_{-2}$ motion $0 \longrightarrow 22$ when R takes {0, 7, 15} to {22, 7, 15} across their shared vertical edge.

2.4. The Toroidal *Tonnetz*

Before navigating the *Tonnetze*, we need to confront their limitation as a representation of pitch-class relations in equal temperament. In Figures 8 and 9, pitch-classes occur in multiple locations, obscuring their equivalence. Alternatively representing each pitch-class at a single location, as in Figure 5a, has the equally pernicious consequence of obscuring adja-

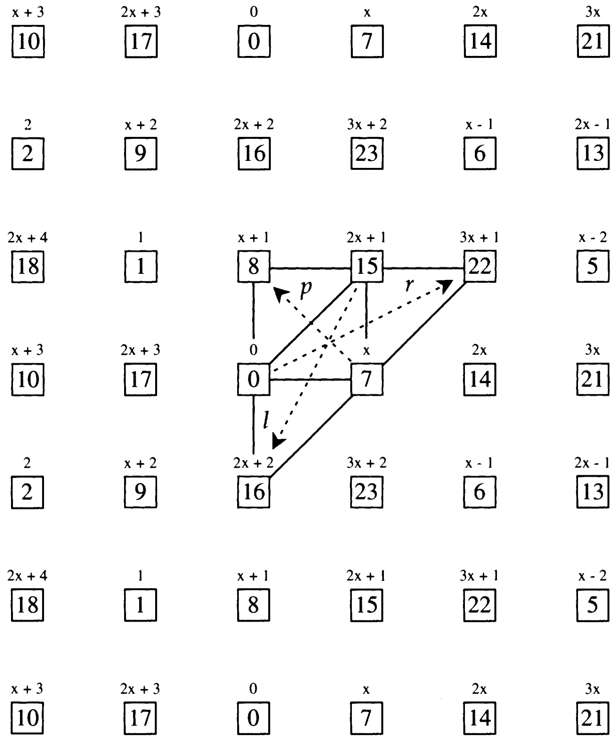


Figure 9: Three Realizations of the Parsimonious *Tonnetz.*
(c) x = 7, y = 8, c = 24, Q' = {0, 7, 15} modulo 24

cency relationships, causing axes to float off the edge of the two-dimensional surface only to reappear on the opposite edge. These obscurities result artificially from the mismatch between the cyclical nature of pitch-class space and the flat surface of the printed page. A torus presents a geometric figure more appropriate to representing the cyclic properties of equal-tempered pitch-class. Although the torus is eschewed here because it is difficult to render and interpret on the two-dimensional surface of the page, its underlying status needs to be sufficiently acknowledged before the *Tonnetz* can be navigated with full comprehension.

At issue above all is the cyclic periodicities of the axes, which fluctuate according to the generating intervals and the cardinality of the chromatic system. The axes relevant to *Tonnetz* navigation are the three that are generated by step-intervals of the parsimonious trichord. These include not only the primary x and y axes, but also the -(x + y) axis that generates the southwest/northeast diagonal. Assigning an abbreviated variable to this step-interval will simplify our treatment of the axis-cycle generated by it.

DEF. (8). $z = c - (x + y)$.

All three step-interval-generated axes are circularized by equal temperament, and thus will be referred to as axis/cycles.

Our study of the periodicity of axis/cycles will be aided by putting into play a variable q*, representing the periodicity of step-interval q modulo c.

DEF. (9). $q^* = \frac{c}{\gcd(c,q)}$, where gcd(c,q), the greatest common divisor of c and q, is the largest integer j such that $\frac{c}{j}$ and $\frac{q}{j}$ are positive integers.

The asterisk is attachable to any variable that represents a step-interval; hence x* is the periodicity of x modulo c, and so forth.

There are two general cases.[17] If q and c are co-prime, then gcd(c,q) = 1, in which case q* = c. The q cycle then exhausts the chromatic system, and all cycles along the q axis are identical; in effect, there is a single q axis-cycle. A familiar example is presented by the z axis of Figure 9a (p. 15), where c = 12 and z = 5. gcd(12,5) = 1, and so z* = c = 12. (The 12-periodicity cannot be directly verified on the attenuated representation of the z-axis given in Figure 9a, but must be induced by extending the boundaries of the figure.) All z axes in Figure 9a thus have a periodicity of 12, and exhaust the 12 pitch-classes via the "circle of fifths." Thus there is only a single distinct z axis-cycle.

In the second general case, gcd(c,q) > 1, in which case q* < c. The q axis does not exhaust the chromatic system, but instead runs through some proper subset of its pcs. The pcs modulo c are then partitioned into $\gcd(c,q) = \frac{c}{q^*}$ co-cycles (provided that 1 is the greatest common divisor of the three step-intervals in Q, as is indeed true for all cases relevant to this study). An example is presented by the z axis of Figure 9c, where c = 24 and z = 9. gcd(24,9) = 3, and so $z^* = \frac{c}{\gcd(c,z)} = 8$: each z cycle includes eight of the 24 pcs in the chromatic system. (Again this claim must be induced from the figure. The ordered content of the z axis beginning at the southwest corner of 9c is as follows: <10, 1, 16, 7, 22, 13, 4, 19, 10>.) There are gcd(24,9) = 3 distinct z axis-cycles, which partition the 24 pitch-classes.

Retreating by one level of abstraction, consider now the cyclic features of Figure 8 (p. 14), where c = 3x + 3, and the three step intervals <x, y, z> are <x, x+1, x+2>. c = 3x + 3 = 3 (x+1) = 3y, and so $y = \frac{c}{3}$. Since y evenly divides c, it follows that gcd(c,y) = y, and so $y^* = \frac{c}{y} = 3$. This explains why there are exactly three distinct elements in each column of the matrices in Figures 8 and 9. The significant point here is that the triple periodicity associated with the second step-interval is proper to the underlying structure of Figure 8, and thus the inclusion of an octave-trisecting interval, acoustically equivalent to a tempered major third, is common to all parsimonious trichords. As for the number of distinct columns, there are gcd(c,y) = y. That is: there is one y-axis co-cycle for each degree of separation between the second and third pitch-class in the prime form of Q′.

In contrast, the remaining step-intervals are divisors of c only under limited conditions. c is co-prime with x (the first step-interval) unless x is a multiple of one of c's divisors, 3 or x + 1. x cannot be a multiple of x + 1; thus c and x are co-prime unless x is an integral multiple of 3, i.e., there is some positive integer n such that 3n = x. If so, then c = 3(3n) + 3 = 9n + 3, in which case c is congruent to 3 modulo 9. The smallest examples of such systems are c = {12 (N.B.), 21, 30}.

Of the systems portrayed in Figure 9, only Figure 9a (p. 15), where c = 12, meets this condition, and consequently it is only here that the x axis partitions its pcs into co-cycles rather than exhausting them in a single cycle. In this case, x = 3, gcd(c,x) = 3, and $x^* = \frac{12}{3} = 4$. Each x axis thus contains four distinct pcs, and there are x = 3 distinct x axes. (In standard terms, of course, what we have here is the partition of the aggregate into three diminished seventh chords.) By contrast, in Figures 9b and 9c, neither c = 18 nor c = 24 are congruent to 3 modulo 9. Consequently, c and x are co-prime, and so there is only a single x axis that exhausts the system of 18 or 24 pcs.

A similar situation holds for the relationship of c to the third step-interval. x + 2 cannot be a multiple of x + 1, and so, in parallel with the case of the x axis just described, c and x + 2 are co-prime unless x + 2 is an integral multiple of 3, i.e., there is some positive integer n such that 3n - 2 = x. In such cases, c = 3(3n - 2) + 3 = 9n - 3. That is, $c \equiv 6$ modulo 9. The smallest such systems are c = {6, 15, 24}. Of the systems portrayed in Figure 9, only Figure 9c, where c = 24, meets this condition, and consequently it is only here that the z axis partitions its pcs into co-cycles rather than exhausting them in a single cycle. By contrast, in Figures 9a and 9b, neither c = 12 nor c = 18 are congruent to 6 modulo 9, and so c and z are co-prime, and there is only a single z axis which exhausts the system of 12 or 18 pcs.

A practical demonstration of the features discussed in this section is provided in the pioneering microtonal treatise of Alois Hába (1927), which systematically explores the generative powers of intervals in both quarter-tone (c = 24) and third-tone (c = 18) systems (where "tone" is taken in the sense of "whole tone"). In his discussion of the quarter-tone system, Hába notes that the "neutral third," equivalent to 7/24 of an octave, generates all 24 pitch-classes. In contrast, Hába writes that the interval "a quarter-tone higher than a major third [i.e. 9/24 of an octave] leads only as far as an octachord [*Achtklang*]. This octachord is composed of the symmetric partition of a major sixth.... Two transpositions of the octachord upward by a quarter-tone use the remaining 16 tones of the quarter-tone-scale" (1927, 166). Hába's three octachords are equivalent to the three distinct z-axis co-cycles discussed above in association with Figure 9c. In a subsequent chapter, Hába approaches the third-tone system from a similar perspective, observing that "the successive series of 5/3 steps forms a unified collection of 18 tones in the third-tone system, and indeed in a more broadly expanded distribution across a span of five octaves. The successive series of 18 seven-third tones forms a collection spread across seven octaves" (202–203). Hába's aggregate-completing *Nacheinanderfolgen* are equivalent to the x and z axes of Figure 9b. The perspective cultivated throughout this section provides a systematic foundation for Hába's observations.

Figure 10 (p. 22) summarizes the work of this section by providing a table for calculating the cyclic periodicities for the chromatic systems that can host parsimonious trichords, as exemplified in the three matrices of Figure 9. The significant point to be carried out of this exposition is that the y axis, generated by the second step-interval, has a constant periodicity, and the number of y co-cycles varies with the size of the chromatic system. Conversely, the x and z axes, generated by the first and third step-intervals respectively, have a variable periodicity, but the number of x and z co-cycles is constant to within the modulo-3 congruence of c. The relevance of these findings, and particularly of the special status of the y axis, to progressions based on PLR-family operations will become apparent in Section 3.

3.1. PLR-family Compounds

We are now in a position to navigate the toroidal *Tonnetz*, in all its various manifestations, using PLR-family operations as our vehicle. The maximal common-tone retention inherent to these operations insures that the cruise will be smooth, particularly when the *Tonnetz* is parsimonious. Our exploration will follow a systematic program, focusing on tri-

(a)	(b)		(c)		(d)		(e)
	x axis		y axis		z axis		
c mod-9	periodicity (x*)	# co-cycles (c/x*)	periodicity (y*)	# co-cycles (c/y*)	periodicity (z*)	# co-cycles (c/z*)	example
0	c	1	3	$\frac{c}{3}$	c	1	c = 18 (Fig. 9b)
3	$\frac{c}{3}$	3	3	$\frac{c}{3}$	c	1	c = 12 (Fig. 9a)
6	c	1	3	$\frac{c}{3}$	$\frac{c}{3}$	3	c = 24 (Fig. 9c)

Figure 10: Cyclic Periodicities of Step Intervals of Parsimonious Trichords

chordal progressions, or chains, generated by the recursive application of a pattern of PLR-family operations.[18] Where appropriate, chains will be viewed as segments of cycles. The basic cycle-classes formed by generated PLR-family chains are few in number, and their relation to PLR-family operations is roughly analogous to the role of the chain of fifths (*Sechterkette*) in diatonic progressions: they constitute normative prototypes against which particular progressions, in all their variety and complexity, may be gauged.

The ultimate goal of this investigation is the pragmatic one of exploring parsimonious voice-leading among consonant triads in a 12-pc system. Consistent with the framework established in Section 2, this familiar phenomenon is situated as a particular manifestation of a more general one: the behavior of parsimonious trichords in any pitch-class system that is suitably sized to host them. Some readers may be frustrated by this strategy, since it defers an encounter with music in systems that we care about, instead inviting contemplation of hypothetical musical systems whose sounds we may have difficulty imagining. Nowadays, of course, the pragmatic fallout of such a study, in the form of "microtonal universes," is readily available to composers, analysts, and listeners. From this viewpoint, the research presented in this paper reflects the deep influence of Gerald Balzano's classic study (Balzano 1980). Like Balzano, my motivations are not only compositional. They stem as well from an intuition, perhaps a credo, that insights into the properties and behavior of individual instances are furnished by studying the properties and behaviors of general phenomena which they represent. As inhabitants of a planet that sustains life, the value of exploring other planets, solar systems, or galaxies for their life-sustaining properties, or lack thereof, potentially transcends the conceivable material benefits, extending to the self-knowledge that emerges from the differentiating context furnished by the Other.[19]

We begin with some notations, definitions, and observations invoked throughout the rest of the paper:

3.1.1. **Notation of Compound Operations**. A compound PLR-family operation is denoted as an ordered set of individual PLR-family operations, enclosed in angled brackets. The operations apply in the order in which they appear in the set, from left to right. For example, in the compound operation <RPL>, R is applied, P is applied to the product of R, and L is applied to the product of R-then-P.

3.1.2. **T/I Equivalences of Compound Operations**. All compound operations are equivalent to either transpositions or inversions, depending on the cardinality of the ordered set. Compound operations of odd cardinality are inversions, those of even cardinality transpositions. This follows from the inversional status of PLR-family operations, together with the group structure of inversion and transposition (see, e.g., Rahn 1980, 52).

3.1.3. **Generators**. A PLR-family compound is **generated** if it can be partitioned into two or more identical ordered subsets. The ordered subset, singly iterated, constitutes the **generator** of the compound. The compound can be expressed as H^n, where H is the generator and n counts its iterations in the compound. A generator of cardinality #H is classified as #H-nary (hence **binary, ternary**, etc.). For example, the compound <PRPRPR> is binary-generated, since it can be partitioned as <<PR> <PR><PR>>, and expressed as $<PR>^3$.

3.1.4. **T/I Equivalences of Generators**. The cardinality of a generated compound H^n is #H · n. If either #H or n are even , then H^n is a transposition. In order for H^n to be an inversion, #H and n must both be odd. This follows from the observations made in 3.1.2 together with the multiplicative properties of odd and even integers.

3.1.5. **Cycles**. H generates a cycle if, operating on some trichord Q, there is some integer q* > 0 such that $H^{q^*}(Q) = Q$. The smallest such value q* is the **operational periodicity** of H. It will also be useful on occasion to count the trichords that result from an H-generated cycle. That number, the **trichordal periodicity** of the cycle, is equivalent to #H · q*.

3.1.6. **Tonnetz Representations of Cycles.** Generators of odd cardinality are involutions: they retreat to their point of origin on the *Tonnetz* after two iterations. This is restated formally as Theorem 2 in the appendix, where a proof is offered. Generators of even cardinality, by contrast, are devolutions: they move perpetually away from their point of origin on the *Tonnetz*. Cycles are generated only when a modular congruence governs the *Tonnetz*, in which case the generator has the same periodicity as the transpositional operation to which it is equivalent (cf. 3.1.4). These periodicities can be computed by first expressing the PLR-family generator as a transposition operation T_n, and then determining the periodicity of n in relation to the size of the chromatic system. For this second step, we will rely on the work carried out in Section 2.4 and summarized in Figure 10.

3.2. Binary Generators

Because the unary generators <L>, <P>, and <R> are involutions, the progressions that they generate are insufficiently varied to serve as compelling musical resources. Thus our exploration begins with binary generators that pair distinct PLR-family operations. There are six such generators, which group into three retrograde-related pairs:

(1) <PR> and <RP>; (2) <LP> and <PL>; (3) <LR> and <RL>.

It has already been determined (cf. 3.1.2) that each binary operation is equivalent to some T_n. Each transpositional value **n** associated with a binary operation is equivalent to a non-zero directed interval within the trichord that is the object of operation. To see why this is so, consider that each individual PLR-family operation alters one pitch-class in Q, and so each binary operation alters two pitch-classes in Q. It follows that the product of a binary operation shares at least one common pc with Q. The common-tone theorem for transposition (Rahn 1980, 108) dictates that if $Q \cap T_n(Q) > 0$, then there is some $\{q_1, q_2\} \in Q$ such that $q_2 - q_1 = n$. From this it follows that each binary PLR-family operation transposes Q by some interval internal to it. The transpositional values associated with the six binary operations are exactly the three step-intervals and their inverses, ±x, ±y, and ±(x + y).

Figure 11 (p. 26) matches directed intervals to their associated binary operations. The six directed intervals of Q are listed as transpositional values at (a). Their binary operation equivalents are shown at (b). The transpositional values are implemented on Q at (c). The binary operations are implemented at (d), where they are represented on the abstract *Tonnetz* as arrows leading out of Q. Each operation transposes Q by one position along the axis representing the step-interval with which it is associated. (For example, the association between T_x and <RP> is confirmed by the matrix position of <RP>(Q), one step rightward of Q along the x axis.) Consequently, the vertices of each resultant triangle at (d) are exactly the pitch-classes resulting from the associated transposition at (c). (For example, {x, 2x, 2x + y} appears both as the set of vertices of <RP>(Q) at (d) and as the result of T_x(Q) at (c).)

Note that inversely related step-intervals are associated with retrograde-related operations. For example, the association of T_x with <RP> is complemented by an association of T_{-x} with <PR>. More generally:

THEOREM 3. For an ordered PLR-family operation set H and its retrograde Ret(H), if H = T_p, then Ret(H) = T_{-p}.

A proof is given in the Appendix.

Moving one step forward into the "middleground," we now examine these relations as they apply to the abstract parsimonious trichordal prime form Q′ = {0, x, 2x + 1}, with directed intervals ±x, ±(x + 1), and ±(x + 2). In general, these intervals represent six distinct values, with the lone exception that, if x = 1 and c = 6, then x + 2 = - (x + 2). This exception aside, the six binary PLR-family operations produce six distinct trichords when implemented on Q′.

Figure 12 (p. 27) translates Figure 11 into terms specific to parsimonious trichords. The transpositions at (a) are given in three forms: as positive and negative generic step-intervals, as positive and negative step-

(a)	(b)	(c)
T_x	<RP>	$\{0, x, x+y\} \xrightarrow{\langle T_x \rangle} \{x, 2x, 2x+y\}$
T_{-x}	<PR>	$\{0, x, x+y\} \xrightarrow{\langle T_{-x} \rangle} \{-x, 0, y\}$
T_y	<PL>	$\{0, x, x+y\} \xrightarrow{\langle T_y \rangle} \{y, x+y, x+2y\}$
T_{-y}	<LP>	$\{0, x, x+y\} \xrightarrow{\langle T_{-y} \rangle} \{-y, x-y, x\}$
T_{x+y}	<RL>	$\{0, x, x+y\} \xrightarrow{\langle T_{x+y} \rangle} \{x+y, 2x+y, 2x+2y\}$
T_{-x-y}	<LR>	$\{0, x, x+y\} \xrightarrow{\langle T_{-x-y} \rangle} \{-x-y, -y, 0\}$

(d)

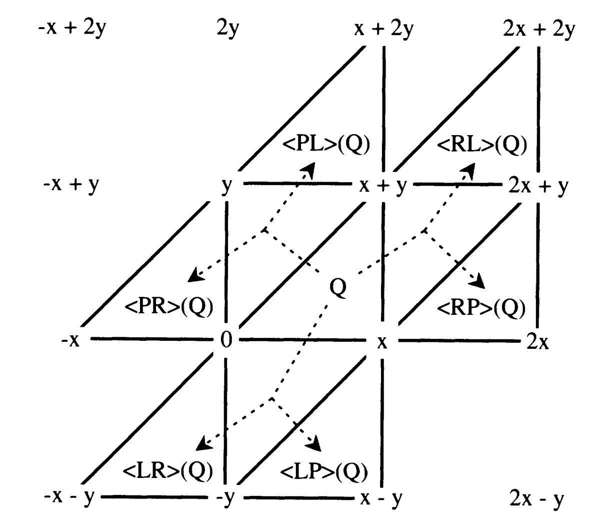


Figure 11: Abstract Transpositional Equivalences
for Binary Generators

(a)			(b)	(c)
T_x			<RP>	$\{0, x, 2x+1\} \xrightarrow{\langle T_x \rangle} \{x, 2x, 3x+1\}$
T_{-x}		T_{2x+3}	<PR>	$\{0, x, 2x+1\} \xrightarrow{\langle T_{2x+3} \rangle} \{2x+3, 0, x+1\}$
T_y	T_{x+1}		<PL>	$\{0, x, 2x+1\} \xrightarrow{\langle T_{x+1} \rangle} \{x+1, 2x+1, 3x+2\}$
T_{-y}	$T_{-(x+1)}$	T_{2x+2}	<LP>	$\{0, x, 2x+1\} \xrightarrow{\langle T_{2x+2} \rangle} \{2x+2, 3x+2, x\}$
T_{x+y}	$T_{-(x+2)}$	T_{2x+1}	<RL>	$\{0, x, 2x+1\} \xrightarrow{\langle T_{2x+1} \rangle} \{2x+1, 3x+1, x-1\}$
T_{-x-y}	T_{x+2}		<LR>	$\{0, x, 2x+1\} \xrightarrow{\langle T_{x+2} \rangle} \{x+2, 2x+2, 0\}$

(d)

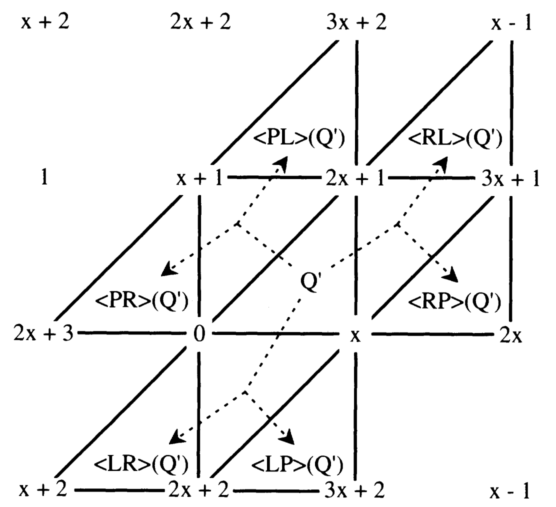


Figure 12: Binary Generators and Parsimonious Trichords

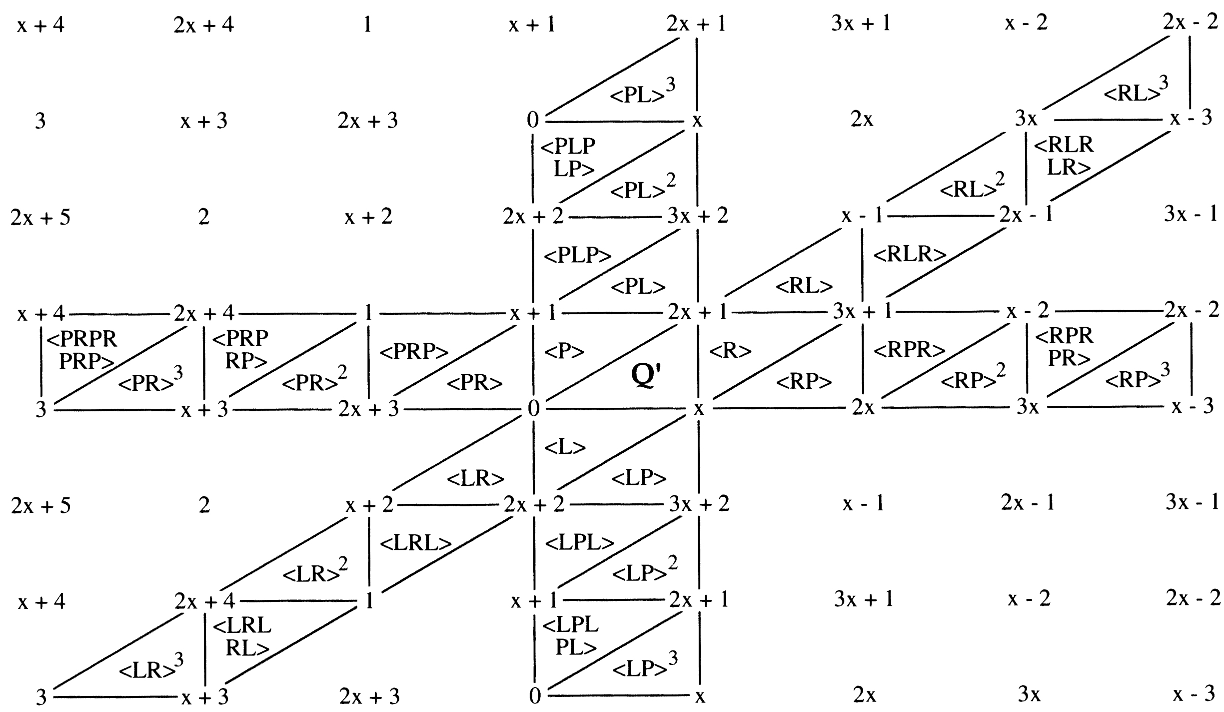


Figure 13: Binary Chains on the Parsimonious *Tonnetz*

intervals of the generic parsimonious trichord Q′, and as positive values modulo 3x + 3. It is these latter values that generate the mappings at (c). Figure 12(d) transfers the label/arrow network from Figure 11(d) onto the parsimonious *Tonnetz* (cf. Figure 8). As with Figure 11, a comparison of the triangle vertices at (d) with the results of the transpositional mappings at (c) confirms the associations of binary PLR-family operation to pc-transposition asserted at (a) and (b).

3.3. Binary Chains and Cycles

Having explored the transpositional behavior of binary operations singly iterated, we now study the chains generated through their recursive application. Figure 13 applies the six generators to Q′= {0, x, 2x+1}, as represented on the parsimonious *Tonnetz* of Figures 8 and 12. Pursuant to Theorem 3, retrograde-related generators proceed inversely out of Q′. Consequently, any generator and its retrograde combine to form a single chain. For the sake of procedural economy, it will be useful to provide a unified label for each chain: invoking alphabetical precedence, the three binary-generated chains will be referred to as <LP>, <LR>, and <PR>.

Each of the three chains represented on Figure 13 threads a space bounded by two parallel and adjacent axes. The <PR> chain, whose generator is associated with the x interval, threads a pair of x axes and thus proceeds horizontally; the <LP> chain, associated with the y interval, threads a pair of y axes and thus proceeds vertically; and the <LR> chain, associated with the x + y = z interval, threads a pair of z axes and thus proceeds diagonally. These affiliations between binary chains, step intervals, and *Tonnetz* directions are constant to all realizations of the generic *Tonnetz*.

In what sense are the chains of Figure 13 cyclic? With the <LP> chain, the cyclicity is readily apparent: both *<PL>*^{3}, at the top of the figure, and *<LP>*^{3}, at its bottom, are equivalent to Q′, indicating that the operational periodicity of the <LP> chain is 3. (Note that this periodicity is proper to the parsimonious *Tonnetz*; were the chains traced on the generic *Tonnetz* of Figures 6 and 11, no such cyclicity would be evident.) With the other two chains, no such equivalences are apparent, and so these chains are not cyclic in the abstract. Once a specific integer value is assigned to x, however, the <LR> and <PR> chains become cyclic. For example, if x = 3, then <PRPRPRP>(Q) = {3, x + 4, 2x + 4} at the figure's left edge is equivalent to <R>(Q) = {x, 2x + 1, 3x + 1} just right of center: both trichords are equivalent to {3, 7, 10}. This indicates that the <PR> chain has an operational periodicity of 4. This periodicity does not hold, however, if x is assigned a different value. It is proper to the Oettingen/Riemann *Tonnetz*, but not to the parsimonious *Tonnetz* of Figure 13, where,

Modulo c		(a) operational periodicity	(b) chain-class cardinality	(c) trichordal periodicity	(d) pitch-class cardinality per co-cycle
<LP>		3	$\frac{c}{3}$	6	6
<PR>	$c \equiv 3$ modulo 9	$\frac{c}{3}$	3	$\frac{2c}{3}$	$\frac{2c}{3}$
	$c \equiv 0, 6$ modulo 9	c	1	$2c$	c
<LR>	$c \equiv 0, 3$ modulo 9	c	1	2c	c
	$c \equiv 6$ modulo 9	$\frac{c}{3}$	3	$\frac{2c}{3}$	$\frac{2c}{3}$

Figure 14: Periodicities and Cardinalities of Binary-Generated Cycles

unlike <LP>, neither the <PR> nor the <LR> chain exhibits constant periodicity.

Although variable, the operational periodicities of <PR> and <LR> for specific values of x can nonetheless be easily determined. To do so, we only need coast on the momentum of our labor from Section 2.4. Because binary operations are equivalent to step-interval transpositions, as discussed in Section 3.2, the two necessarily have identical periodicities. We have already seen this for the case of the <LP> chain, whose operational periodicity of 3 derives from the triple periodicity of its associated step-interval y (cf. Figure 10(c), p. 22). The periodicities for <PR> and <LR> likewise transfer from that of their associated step-intervals, x and z, respectively. Figure 10(b) gives the value of x*, the periodicity of x modulo c, and these values apply directly to the periodicity of <PR>: if $c \equiv \{0,6\}$ modulo 9, then the operational periodicity of <PR> is c; if $c \equiv 3$ modulo 9, the periodicity is $\frac{c}{3}$. Figure 10(d) gives the value of z*, the periodicity of z modulo c, and these values apply directly to the periodicity of <LR>: if $c \equiv \{0,3\}$ modulo 9, then the periodicity of <PR> is c; if $c \equiv 6$ modulo 9, the periodicity is $\frac{c}{3}$. These periodicities are summarized in column (a) of Figure 14.

Other significant features of binary-generated cycles are directly inferrable from the foregoing.

Cardinality of cycle-classes. When one of the chains on Figure 13 (p. 28) is transposed, axes and thus trichords may be exchanged, but the class membership of the chain (<LP>, <PR>, or <LR>) is preserved. It is thus important to make an ontological distinction between specific cycles and the classes that they represent. This distinction leads us to ask how many distinct cycles belong to each class, i.e. what is the cardinality of that class. To answer this question, we begin by observing that each cycle engages two pitch-class axes, and each axis participates in two cycles. It follows that the cardinality of each binary-generated cycle-class is equivalent to the quantity of distinct pitch-class axes of that generator's step-interval associate, as given in Figure 10 (p. 22). This quantity equals $\frac{c}{q^*}$, where c measures the chromatic system as in Def. (1), and q* is the periodicity of the binary generator as at Figure 14, column (a). This information is summarized in Figure 14, column (b).

Trichordal periodicity. The trichordal periodicity of a generated cycle equals the product of operational periodicity times generator cardinality (cf. 3.1.5). In the case of binary generators, trichordal periodicity doubles operational periodicity. All <LP> cycles thus engage six trichords, but the trichordal periodicity is variable for the <PR> and <LR> cycles. This information is summarized in Figure 14, column (c) for each cycle-class.

Pitch-class engagement. Consider a binary operator H associated with step-interval q, where q* represents the periodicity of q and hence of H.

- If q and c are co-prime, q* = c (cf. 2.4). Then both axes threaded by an H-generated chain include the entire pc-aggregate. It follows that an H-generated chain engages all c pitch-classes of the chromatic system.
- If, however, q and c share a common divisor greater than 1, then q* < c, and the c pitch-classes partition into co-cycles (cf. 2.4). The two q axes threaded by an H-generated chain thus have distinct pc-content. Since each of these axes has a periodicity of q*, it follows that an H-generated chain engages 2q* pitch-classes. Thus an <LP> cycle always engages a hexachord. (The prime form of this hexachord is {0, 1, x + 1, x + 2, 2x + 2, 2x + 3}, with step-intervals <1, x, 1, x , 1, x>.) Depending on the modulo 9 congruence of c, the <LR> and <PR> cycles either engage the entire pitch-class system, or $\frac{2}{3}$ of that system. (In the latter case, the prime form of the set engaged by the cycle can be represented as {«3n, 3n + 1«} for n := 0 to $\frac{c}{3}$, with step intervals <1, 2 1, 2...>.) This information is summarized in Figure 14, column (d).

(a)		(b)	(c)	
T_x	T_3	<RP>	$\{0, 3, 7\} \xrightarrow{<T_3>} \{3, 6, 10\}$	$C- \xrightarrow{<T_3>} Eb-$
T_{-x}	T_9	<PR>	$\{0, 3, 7\} \xrightarrow{<T_9>} \{9, 0, 4\}$	$C- \xrightarrow{<T_9>} A-$
T_y	T_4	<PL>	$\{0, 3, 7\} \xrightarrow{<T_4>} \{4, 7, 11\}$	$C- \xrightarrow{<T_4>} E-$
T_{-y}	T_8	<LP>	$\{0, 3, 7\} \xrightarrow{<T_8>} \{8, 11, 3\}$	$C- \xrightarrow{<T_8>} Ab-$
T_{x+y}	T_7	<RL>	$\{0, 3, 7\} \xrightarrow{<T_7>} \{7, 10, 2\}$	$C- \xrightarrow{<T_7>} G-$
T_{-x-y}	T_5	<LR>	$\{0, 3, 7\} \xrightarrow{<T_5>} \{5, 8, 0\}$	$C- \xrightarrow{<T_5>} F-$

(d)

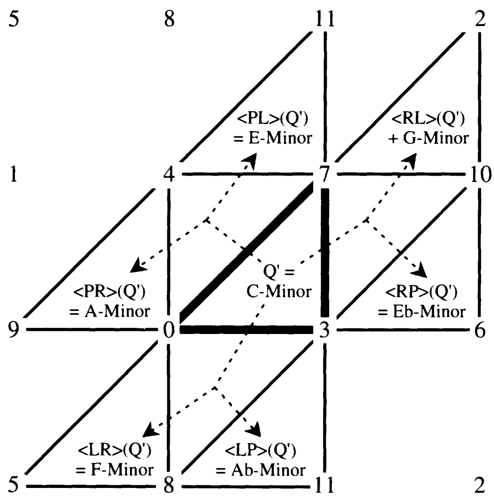


Figure 15: Binary Generators and Consonant Triads modulo 12

In general, the smaller the periodicity of the step interval associated with an operation, the more economical is the pitch-class design of the cycle generated from that operation. The significance of this circumstance for nineteenth-century harmony will become clear in Section 3.4.

3.4 Binary Chains and Cycles in Modulo 12

In this section, the properties of parsimonious trichords established above are studied as they apply to the object of historical and analytical interest, the consonant triad in a system of 12 pitch-classes. We begin by studying the six binary operations, singly iterated, as we did in association with Figure 12 (p. 27). Figure 15 provides a realization of Figure 12, with x = 3. This assignment immediately brings the familiar object and system into view. $Q' = \{0, x, 2x + 1\}$ is realized as the consonant triad $\{0, 3, 7\}$ = C minor, the step intervals <x, x + 1, x+ 2> as <3, 4, 5>, and the chromatic system 3x + 3 as the 12-pc system. The parsimonious *Tonnetz* becomes an equal-tempered version of the one discovered by Euler and Oettingen and propagated by Riemann (cf. Figure 9a). The triads produced by the six binary operations indicated on Figure 15 constitute a complete inventory of the minor triads that share a pitch-class with C minor.

To explore the recursive properties of these operations, Figure 16 (p. 34) transfers the three binary-generated chains of Figure 13 onto the Oettingen/Riemann *Tonnetz*. (These chains are somewhat re-positioned, in part for visual clarity.) Each of these three classes of triadic progression is familiar from music of the nineteenth century, and each has distinctive properties.

The group structure of <LP> chains has been studied by Hyer (1989, 1995), and some of their special qualities were explored from different perspectives in an earlier paper of mine (Cohn 1996, where I refer to a chain of this type as a "maximally smooth cycle") and in Lewin (1996, who classifies it as a "Cohn Cycle"). The particular <LP> chain threading the vertical axes of Figure 16 is the one that is traversed in a downward direction in a passage from Brahms's Double Concerto that is modelled in Example 1 and that I have already studied in some detail (Cohn 1996, 13–15). The matrix representation illustrates several significant features of <LP> cycles identified in my earlier paper:

Set-class Non-Exhaustion. <LP> generates a cyclic progression of 6 triads, a sub-group of the 24 triads.

Limited pc engagement. Both engaged columns have a cyclic periodicity of 3, so that an <LP> cycle engages only 6 pitch-classes. The pc-set unites two adjacent IC-4 cycles (represented by the two y-axis columns) into a hexatonic set belonging to Forte-class 6-20.[20]

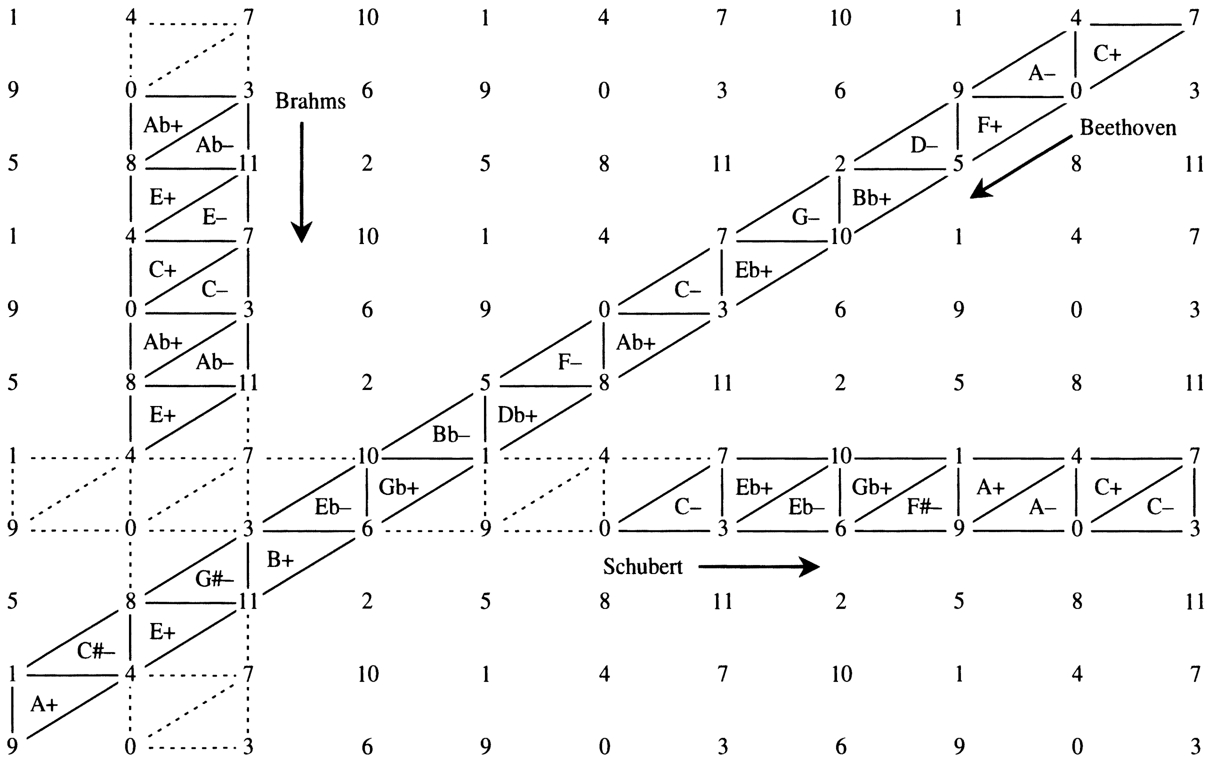
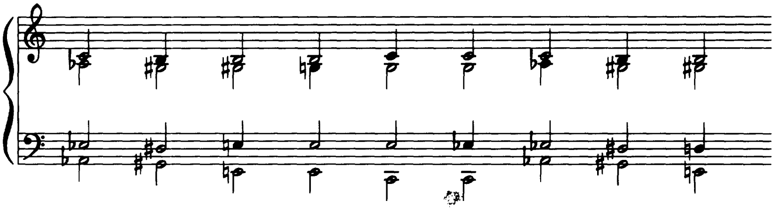


Figure 16: Binary Chains on the Oettingen / Riemann *Tonnetz*



Example 1: Brahms, Concerto for Violin and Cello, Op. 102, First Movement, mm. 270–78

Example 2: Schubert, Overture to *Die Zauberharfe*, opening *Andante*

Multiple distinct cycles. The number of distinct <LP> cycles is equivalent to the number of distinct y axes: four. In my earlier paper, I refer to these four cycles as **hexatonic systems**, in reference to their limited pc-content. In the current context they are more appropriately referred to as sub-systems.

The particular **<PR> cycle** threading the horizontal axes of Figure 16 is the one that is traversed right-ward (in reference to the spatial layout of Figure 16) in Example 2, which models the Andante introduction of the overture to Schubert's opera *Die Zauberharfe* (1820).[21] The three features proper to <PR> cycles are analogous to those identified above for <LP> cycles, and this analogy is reflected in the parallel description of them below:

Set-class Non-Exhaustion. <PR> generates a cyclic progression of 8 triads, a sub-group of the 24 triads.

Limited pc engagement. Both engaged rows have a cyclic periodicity of 4, so that a <PR> cycle engages 8 pitch-classes. The pc-set unites two adjacent IC-3 cycles (represented by the two x-axis rows) into an octatonic set belonging to Forte-class 8-28.

Multiple distinct cycles. The number of distinct <PR> cycles is equivalent to the number of distinct x-axes: three. These cycles can be referred to as **octatonic sub-systems** (cf. Lerdahl 1994, 132–33). Any <PR> cycle-segment is a (proper or improper) subset of one of the three <PR> cycles.

The situation with the **<LR> cycle,** which threads the diagonal axes of Figure 16, is entirely different. The three features noted for the <LP> and <PR> cycles have no analogue here.

Set-class Exhaustion. The compound operation <LR> generates the entire group of 24 triads.
PC-Aggregate Completion. Both engaged z-axis diagonals have a cyclic periodicity of c = 12, and an <LR> chain engages all 12 pitch-classes;
One Single Cycle. There is a single <LR> cycle, of which any <LR> cycle-segment is a subset.

The complete <LR> cycle is too long to sustain compositional interest under normal conditions. This is brought out by a thirty-four-measure passage which I have written about on two earlier occasions (Cohn 1991 and 1992) from the second movement of Beethoven's Ninth Symphony (mm. 143–76) . From a starting point at C-major in the northeast corner of Figure 16, the passage obsessively whirls through $<RL>^9$, traversing eighteen triads before halting at the nineteenth link, the A-major triad in the southwest corner. Even a squall of such velocity and force cannot propel itself about the entire cycle.

Nonetheless, the <LR> cycle has a more venerable legacy than the other binary-generated cycles. Singly iterated, <LR> transposes by perfect fifth, the transpositional value that preserves maximum pc-intersection between diatonic collections (as well as the interval that sports acoustic privilege). The <LR>-cycle was initially recognized in thorough-bass methods from the late seventeenth century as a gauge of modulatory distance (Lester 1992, 215). In 1827, Johann Bernhard Logier recommended that young musicians learn to play the entire cycle at the pianoforte, "as it forms the groundwork on which may be constructed an almost infinite number of passages and variations." Logier's assessment of the cycle's properties is pertinent to the approach adopted here:

> We perceive, from the beginning to the end, not only a beautiful symmetry and regularity pervading the whole, but also a *double* union of intervals—two of them always remaining undisturbed....the whole progression forming a chain of harmony unequalled in any of our former exercises.[22]

As these passages suggest, the entire cycle is less a model of a surface structure to be traversed in a single gesture than a compositional space which, like a city, becomes entirely known through an exertion of memory across a set of partial explorations. In a more recent formulation directly related to the approach developed in this paper, Lewin (1987, 180) notes that the 24 triads are generable by a single operation, MED,

c = 12 ≡ 3 modulo–9	(a) operational periodicity	(b) chain-class cardinality	(c) trichordal periodicity	(d) pitch-class cardinality per co-cycle
<LP>	3	4	6	6
<PR>	4	3	8	8
<LR>	12	1	24	12

Figure 17: Binary-Generated Periodicities and Cardinalities Modulo 12

which maps a triad to its diatonic submediant. The powers of MED organize the 24 triads into a simply transitive group, a Generalized Interval System isomorphic with the <LR> cycle. In this sense, the <LR> cycle can be seen as a source for all triadic progressions. The <PR> and <LP> cycles emerge, among other routines, as patterned sub-groups of the <LR> cycle.[23]

Figure 17, modelled after Figure 14, summarizes the properties of the binary-generated cycles modulo 12. Surveying the three cycles as an ensemble, what is most striking is the affinity between the <LP> (hexatonic) and <PR> (octatonic) cycles, whose periodicites and cardinalities are nearly identical, and the anomalous status of the <LR> cycle by the same standard. These affiliations are unexpected in the general context provided by Section 3.3, which emphasized the affinities of the <PR> and <LR> cycles, on the basis of their variable periodicities, and the anomalous status of the <LP> cycle, on the basis of its uniquely constant periodicity. The section that now follows will suggest that the strong affinity between the <LP> and <PR> cycles in modulo 12 is accidental rather than inherent. In no other chromatic system is the affinity so strong.

3.5. Hexatonic and Octatonic Analogues

We begin by investigating the relatively simple case of <LP> cycles in systems other than modulo 12. In each pitch-class system that hosts parsimonious trichords, <LP> cycles have a constant operational periodicity of 3, trichordal periodicity of 6, and pitch-class cardinality of 6 (cf. Figure 14, p. 30). Thus <LP> cycles are inherently hexatonic. The variable

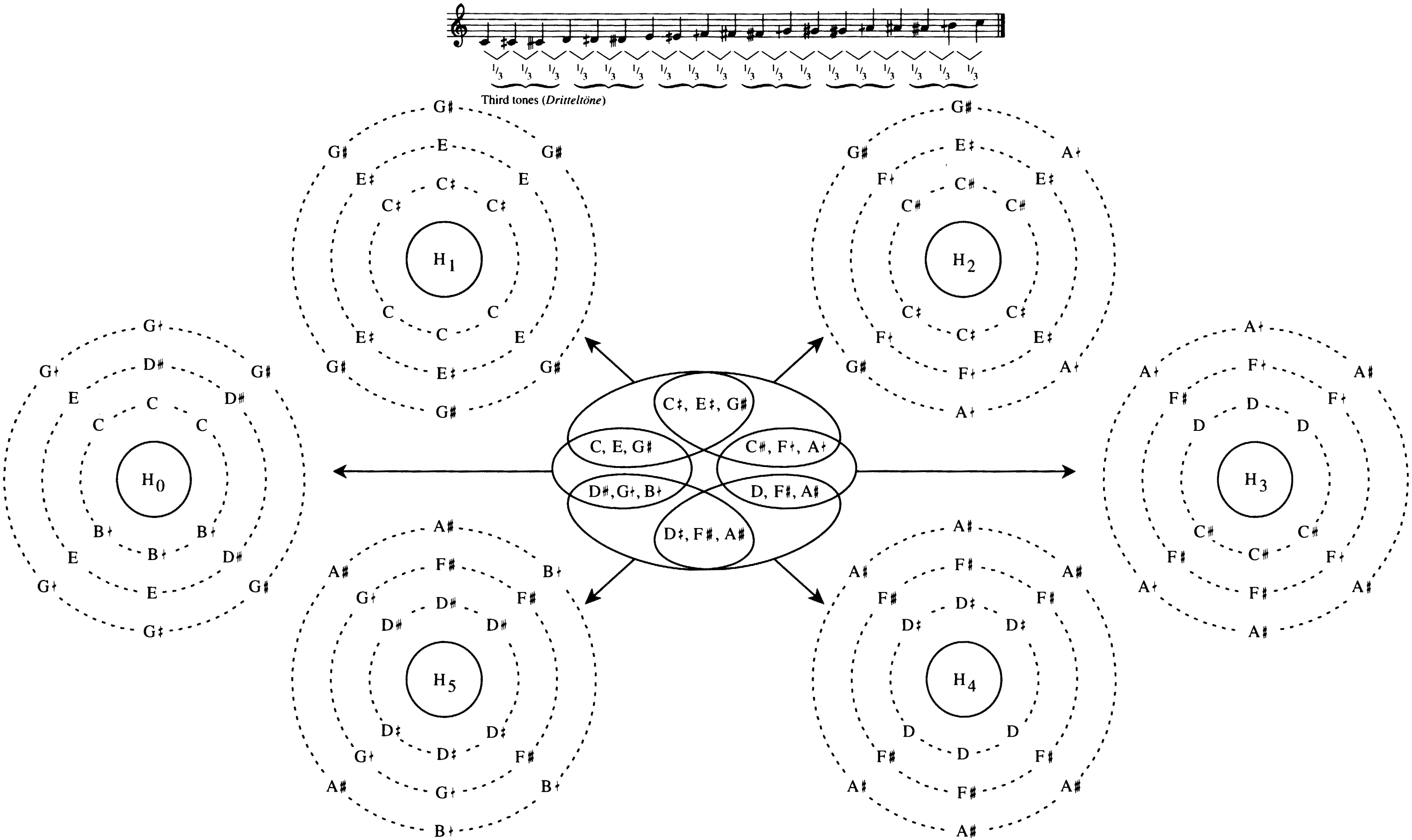


Figure 18: The Hyper-Hexatonic-Analogue System for c = 18

property is the cardinality of the <LP> cycle-class, i.e. the number of distinct hexatonic sub-systems. The 2c parsimonious trichords in a system of c pitch-classes partition into $\frac{2c}{6} = \frac{c}{3}$ <LP> co-cycles. This yields four hexatonic sub-systems for c = 12, six sub-systems for c = 18, eight sub-systems for c = 24, and so forth.

In my earlier paper, which studied the relationship among the four hexatonic sub-systems in the mod-12 system, I observed (Cohn 1996, 23–25) that each sub-system shares pitch-classes with two other sub-systems, and is pitch-class complementary to the remaining sub-system. On this basis, the ensemble of sub-systems is shaped into a four-element system at a higher level. I proposed a two-tier design, where both the six-element sub-systems and the four-element "hyper-system" that embeds them are Generalized Interval Systems (GIS) (Lewin 1987). Our current work suggests that the same two-tiered GIS design applies to <LP> cycles in all chromatic systems where they occur. The individual sub-systems, which I shall call **hexatonic analogues**, constitute GIS's whose elements are the six trichords of an <LP> cycle, and thus whose size is constant to all chromatic systems. The higher-level **hyper-hexatonic analogue** system, organized by pc-intersection among the sub-systems, constitutes a GIS whose size of $\frac{c}{3}$ sub-systems varies with the chromatic cardinality.

Figure 18 demonstrates the hyper-hexatonic-analogue system for c = 18, using Hába's *Dritteltonsystem* pitch-class designations, which are inventoried at the top of the figure. (Hába uses ⸸ for 1/6 sharp, ‡ for 1/3 sharp, ⧺ for 2/3 sharp, and ♯ for 5/6 sharp.) The 18 pitch-classes are arranged into 6 T_6-cycles at the center of the figure. Each such cycle trisects the octave, and is thus acoustically equivalent to a tempered augmented triad modulo 12. The cycles are paired into six overlapping ovals which portray the six hexatonic-analogue pc sets. Each set is connected by an arrow to the hexatonic analogue sub-system of trichords for which it furnishes a pitch-class source. Each such sub-system is an <LP> cycle modulo 18, and each is connected directly, by shared "augmented triad," to two neighboring systems. A unique feature of this chromatic system is that the hexatonic sub-systems and the hyper-system share a cardinality of 6. This isomorphism, which arises from the equality of the variable $\frac{c}{3}$ to the constant 6 when c = 18, presents some interesting compositional possibilities which I shall not explore in the current context.

The <PR> (octatonic) sub-systems of modulo 12 generalize in a quite different way. Whereas proper <LP> sub-systems appear in all suitably sized chromatic systems, proper <PR> sub-systems appear only in those systems where $c \equiv 3$ modulo 9. Even in these systems, the <PR> sub-systems generalize differently than the <LP>. We have seen that <LP> sub-systems have a constant size but a variable quantity (cf. Figure 14). The

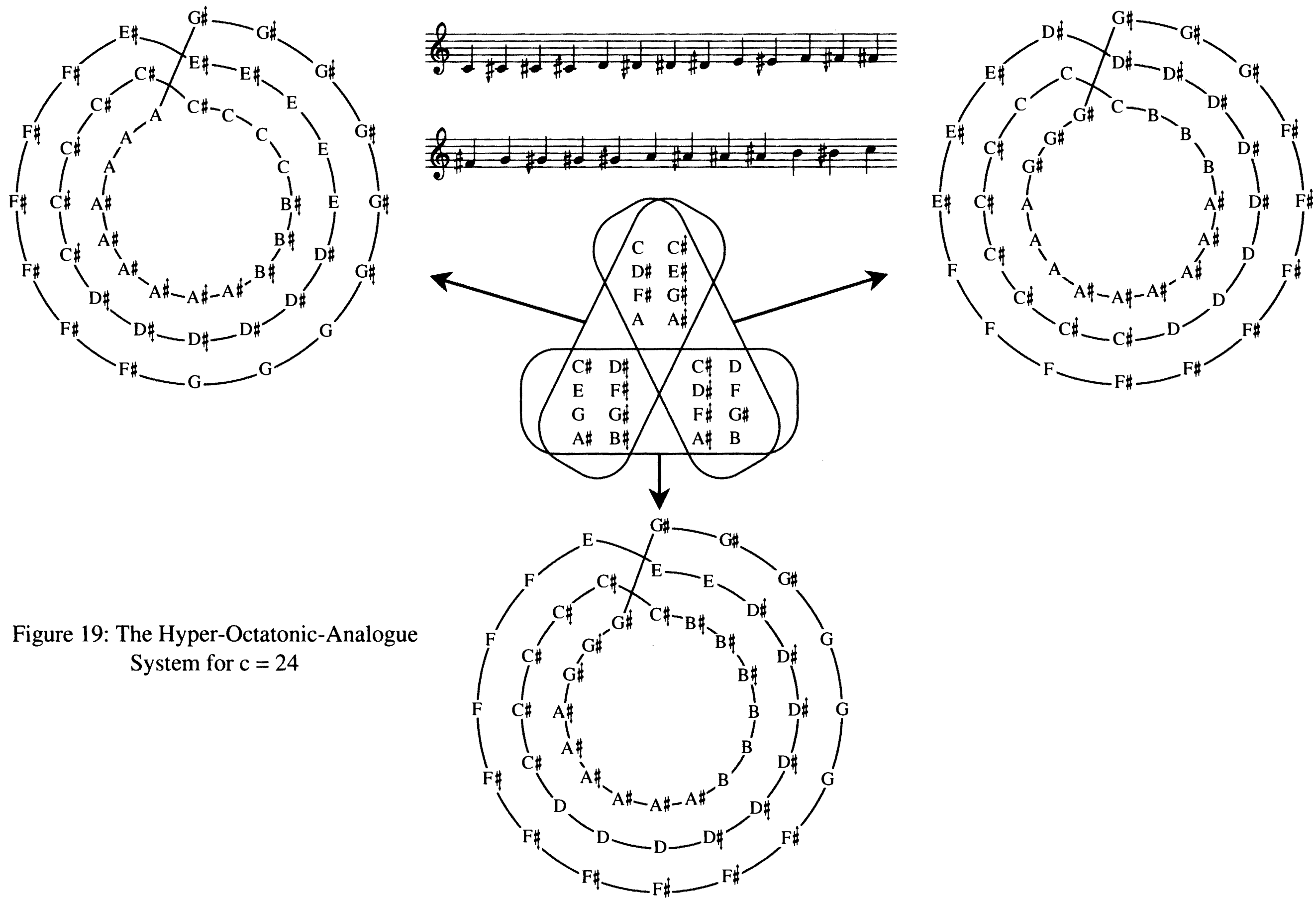


Figure 19: The Hyper-Octatonic-Analogue System for c = 24

situation with the <PR> sub-systems is converse. The quantity of sub-systems is constant: the 2c parsimonious trichords of each system partition into three <PR> sub-systems. The size of each <PR> sub-system is variable, engaging $\frac{2c}{3}$ trichords. Consequently, generalized <PR> sub-systems are not "octatonic" in the same sense that generalized <LP> sub-systems are hexatonic: they have a pc-cardinality of 8 only in case c = 12. The most salient property of the mod-12 octatonic that generalizes to modulo c is the step-interval pattern <1, 2, 1, 2...>. In this sense the term "1:2-analogue" (based on Lendvai 1971) would better accommodate the basis of the analogy than "octatonic analogue." I nonetheless retain the latter term in order to stress the positive aspects of the meta-analogy with "hexatonic analogue."

Although, as we have seen, the <LR> cycle yields no proper sub-systems modulo 12, it does yield proper sub-systems in chromatic systems congruent to 6 modulo 9, such as c = 15, 24, and so forth. In these cases, the <LR> sub-systems are structured in the same way as the <PR> sub-systems, in the sense that there are three sub-systems, each with a trichordal periodicity and pitch-class cardinality of $\frac{2c}{3}$. Because of this similar structuring, it makes sense to apply the term **octatonic analogue** to the <LR> as well as the <PR> sub-systems.

Figure 19 demonstrates the hyper-octatonic-analogue system for c = 24, using Hába's *Vierteltonsystem* pitch-class designations, inventoried in ascending order at the top of the figure. The 24 pitch-classes are partitioned into three T_3-cycles of cardinality 8 at the center of the figure, which are paired by the ovals into octatonic-analogue collections of cardinality 16. These collections furnish the pc content for the octatonic-analogue sub-systems of triads, equivalent to the <LR>-cycles, modulo 24.

c = 24, then, is an example of a system whose 2c parsimonious trichords partition into both hexatonic- and octatonic-analogue sub-systems. The two sub-system classes are equivalent to two of the three binary-generated cycles, <LP> and (in this case) <LR>. The third binary-generated cycle, <PR> in this case, generates the entire set-class of 48 parsimonious trichords, and thus its role in the c = 24 system is analogous to that described in Section 3.4 for the <LR>-cycle in the c = 12 system.

Figure 20 (p. 42), which is cut from the same template as Figures 14 and 17, summarizes the periodicities for a mod-24 quarter-tone system. The affinity between the <LP> and <PR> cycles of modulo 12 has been replaced by an affinity between <LP> and <LR>. Both feature attenuated periodicities and non-exhaustion of the trichord class and the pitch-class aggregate. But the degree of affinity is greatly weakened, as the periodicities and cardinalities diverge. This circumstance underlines the special nature of modulo 12 suggested at the end of Section 3.4. In no other chro-

c = 24 ≡ 6 modulo–9	(a) operational periodicity	(b) chain-class cardinality	(c) trichordal periodicity	(d) pitch-class cardinality per co-cycle
<LP>	3	8	6	6
<PR>	24	1	48	24
<LR>	8	3	16	16

Figure 20: Binary-Generated Periodicities and Cardinalities Modulo 24

matic system does the variable operational periodicity $\frac{c}{3}$ so closely approximate the constant periodicity 3. (Both c = 9 and c = 6 are deficient: c = 9 produces no octatonic-analogue sub-systems because 9 is congruent to 0 modulo 9; although c = 6 produces octatonic analogue sub-systems under <LR>, its two (!) hexatonic sub-systems are not pitch-class distinct.)

I imagine the hexatonic and octatonic sub-systems as objects in space, one fixed and one transient. As the transient object momentarily passes by the fixed one, their relationship becomes recorded, frozen in time like the objects in a photograph. An observer who knows the objects only through the recording is in no position to understand that their association is anything other than permanent and intrinsic. Our study of this phenomenon in the context of generalized chromatic systems unfreezes the moment, allowing us to see the accidental nature of the relationship, and thereby enhances our appreciation of the over-determined qualities of the integer 12.

This concludes the inventory of binary operations and the trichordal progressions that they generate. We now turn our attention to ternary generators, which are of a different nature because they are inversions rather than transpositions.

3.6. Ternary Generators and LPR Loops

We begin our study of ternary generators by demonstrating that, once certain rules of reduction are invoked, their numbers are constrained. The reduction process is facilitated by the concept of **generator form.** Generators that have identical order-position equivalences will be said to share an abstract **form.** Forms are designated by assigning the abstract tokens A, B, and C to the three distinct PLR-family operations in the order that they appear in the generator. Accordingly <LPLR> and <RPRL> are both of the form <ABAC>.

Because two successive iterations of a single operation "undo" each other, ternary generators of the form <ABB> reduce to unary generators (<A(BB)> = <A>), and accordingly have no independent interest. Neither do generators of the form <ABA>, since reiteration juxtaposes the two A's, triggering a chain of involutions that unravels the entire structure: <ABA>^{2} = <AB(AA)BA> = <A(BB)A> = <AA> = T_0. Consequently, independent ternary generators are limited to the form <ABC>. The six realizations of this form correspond to the six orderings of the unordered set {L,P,R}. These six orderings are equivalent under retrograde and rotation. We will view them as members of a single class of ternary chain whose canonical label, invoking alphabetic precedence, is <LPR>.

To illustrate the involutional nature of ternary generators (cf. Theorem 2), Figure 21 (p. 44) models a cycle of six triads on the Oettingen/Riemann *Tonnetz*. Selecting F minor as a starting point, we move clockwise through a semi- cycle, engaging R, P, and L in that order, and stopping at E major. Continuing clockwise from this point, the same three operations are engaged in the same fixed order, closing the cycle back to F minor. The compound operation represents <RPL>^{2}. Regardless of starting triad or direction, the same set of triads is traversed, demonstrating the equivalence of the six orderings of {L, P, R}. I will refer to such structures generically as **LPR loops.**

Two examples of LPR loops from nineteenth-century opera, both using the specific triads included in Figure 21, are presented in Examples 3 and 4 (p. 45). Example 3 models the succession of triads in the cantabile section of "Ah sì, ben mio," from Act III of Verdi's *Il Trovatore*. Beginning in F minor, the first quatrain closes by tonicizing its relative major. The aria's second quatrain "mutates" to A♭ minor, prolongs F♭ major throughout its consequent phrase, repeats the text of the consequent phrase over a D♭ minor triad, and reaches a fermata over a V7 of D♭. The final quatrain of the cantabile then prolongs D♭ major. The six consonant triads traverse an LPR loop, although the loop does not close with a return to the initial F minor.

Example 4 is the *Engelmotiv* from Wagner's *Parsifal*, in a transposition that occurs in Amfortas's Prayer from Act III. The passage begins and ends on D♭ major, and rotates clockwise through the triads of Figure 21, omitting A♭ minor and D♭ minor. As the analysis beneath Example 4 indicates, the omissions are accounted for by compound operations that "elide across" the omitted triads.[24]

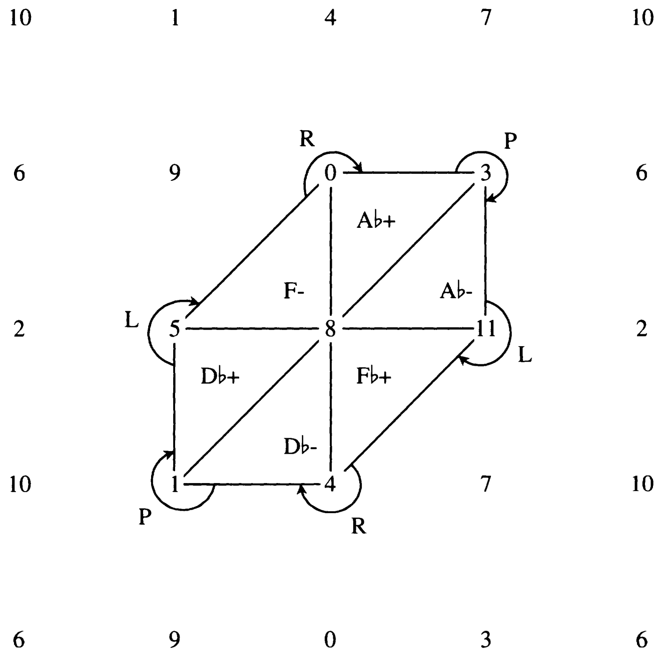
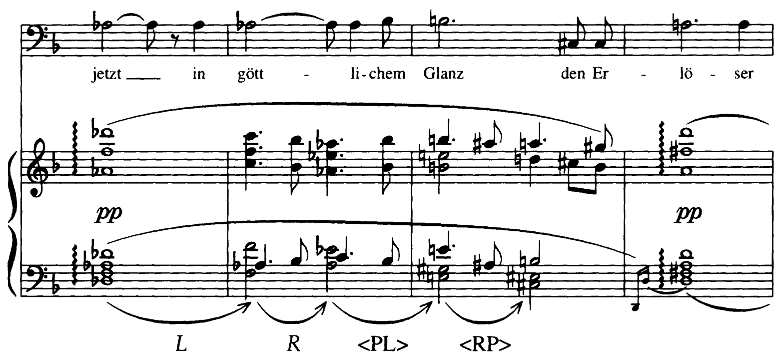


Figure 21: LPR-loop around A♭/G♯

A significant feature of LPR loops is that their six triads share a single pitch-class, located at the center of the matrix representation. In Figure 21, A♭/G♯ plays that role. The members of a loop furnish a complete roster of the triads that include the shared pitch-class.[25] It follows that there are twelve distinct LPR loops (one per pc), that each triad participates in three such loops, and that the three interlocking loops in which that triad participates furnish a complete roster of the triads with which it shares one or more pitch-classes. Figure 22 (p. 46) illustrates for the case of F minor. The figure is in the form of an interlocked set of "honeycombs." The circle at the center of each loop encloses the pitch-class common to all the triads in that loop. The entire structure of the twelve interlocked loops can be easily inferred by projecting outward from Figure 22.[26]

Because of their common-tone properties, LPR loops furnish an ideal progression for consonantly supporting a single melodic pitch with diverse harmony while maximizing voice-leading parsimony. Such progressions, with their implication of inner action or turmoil beneath a

Example 3: Verdi, *Il Trovatore,* Act III: *Ah sì b'en mio*



Example 4: Wagner, *Parsifal,* Act III: *Engelmotiv*

placid and harmonious surface, were well suited to symbolize nineteenth-century notions about the relationship of the inner and outer worlds. Examples include the openings of Liszt's "Il Penseroso" (*Années de Pèlerinage, Deuxième Année,* composed 1839), and of the Monk's Chorus from Verdi's *Don Carlos.*

All of the LPR-loop properties identified here for mod-12 consonant triads also hold, mutatis mutandis, for parsimonious trichords in other systems, as can be seen by transferring the cyclic design of Figure 21 onto the background grid first introduced in Figure 8 (p. 14). Furthermore, analogous properties hold for any trichord in any system, minus the parsimonious voice-leading, as can be seen by transferring the same cyclic design on the background grid of Figure 6 (p. 10).[27] This transferability results from the odd cardinality of ternary generators, which causes them to involute, in contrast to the devolutionary nature (cf. 3.1.6) of binary generators, and of the quaternary generators that we now study.

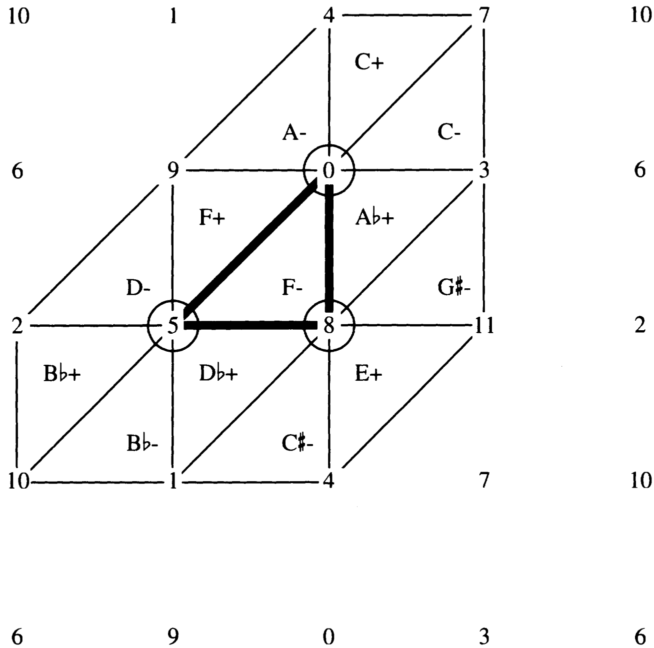


Figure 22: Interlocking LPR-Loops

3.7 Quaternary Generators

Like ternary generators, quaternary generators reduce to a single form once operations that involute (either directly or across the boundaries of successive iterations) are eliminated and rotational equivalence is invoked. Forms with two sets of duplicate operations are eliminated either because the duplicates are juxtaposed and thus undo each other (<A(BB)A> = <AA> = T_0 and <(AA)(BB)> = T_0), or because the generator is itself binary-generated (<ABAB> = <AB>^{2}). Thus, all independent quaternary generators include all three distinct operations, with a single operation represented twice. The two iterations of the duplicate operation can be neither adjacent (as in <(AA)BC>) nor maximally separated (as in <ABCA>, whose reiteration causes <ABC(AA)BCA>), and

(a)	(b)	(c)	(d)	(e)	(f)
L	<LPLR>	<LP><LR>	$T_{-y}\ T_{-(x+y)}$	$T_{-(2y+x)}$	T_{l}
P	<PLPR>	<PL><PR>	$T_{y}\ T_{-x}$	T_{y-x}	T_{p}
R	<RLRP>	<RL><RP>	$T_{x+y}\ T_{x}$	T_{2x+y}	T_{r}

Figure 23: Transpositional Equivalences of Quaternary Generators

accordingly must be separated by two order positions. All quaternary generators that fit this description are equivalent to <ABAC> via rotation. The six generators that realize <ABAC> group into three rotation-related pairs, as follows:

(1) <LPLR> and <LRLP>; (2) <PLPR> and <PRPL>; (3) <RLRP> and <RPRL>.

A significant property of any quaternary generator is that the transposition that it enacts is equivalent to the voice-leading interval produced by the moving voice when a single iteration of the duplicate operation is executed. For example, an operation with duplicate R, such as <RPRL>, is equivalent to T_{r} . Thus **a quaternary operation unfolds in the large the transpositional operation that its duplicate operation expresses in the small.**

Figure 23 demonstrates this "exfoliation" property. The three operations, listed at (a), are represented as duplicates by a quaternary operation at (b). These are each partitioned into a pair of binary operations at (c), and converted to their associated T_n values at (d) (cf. Figure 11, p. 26). The T_n values at (d) are composed at (e) by summing sub-scripts, and the product is expressed in terms of a voice-leading interval at (f), following the equivalences established at Def. (5) in Section 1. A comparison of (f) with (a) demonstrates the correspondence of the transpositional value with the duplicate operation. Note that replacing an operation at (b) with its rotation (e.g. <LRLP> for <LPLR>) does not affect the result, since transposition operators commute and thus <LP><LR> = <LR><LP> (cf. Kopp 1995, 272–73).

Figure 24a uses the the abstract *Tonnetz* (cf. Figure 6) to portray the six quaternary operations as they act on the generic prime-form trichord Q =

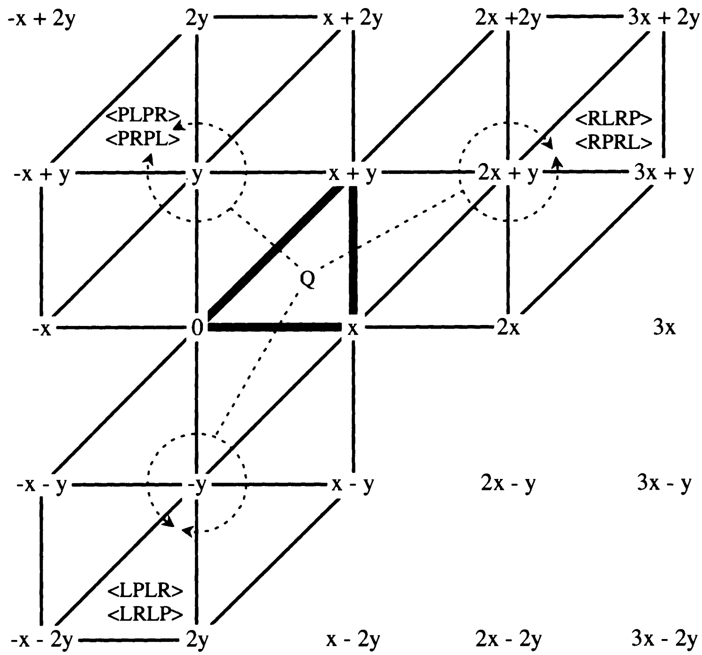


Figure 24a: Quaternary Generators on the Abstract *Tonnetz*

{0, x, x + y} with step-intervals <x, y, -(x+y)>. Each of the three paths leading out of Q implements the first operation of the quaternary set, traversing one of the three edges. The paths then bifurcate, reuniting at the point that the operation is completed. For example, <PLPR> and <PRPL> both start along the northwest path out of Q, traversing the hypotenuse to P(Q). <PLPR> turns right and proceeds counter-clockwise, while <PRPL> forks left and proceeds clockwise. Both operations terminate at $T_p(Q) = T_{y-x}(Q)$ at the northwest corner of the figure.

Figure 24b transfers the same design onto the Oettingen/Riemann *Tonnetz*, where $p = 1$, $l = 1$, and $r = -2$. These voice-leading intervals are echoed as transpositional values:

- To the south of Q′, the duplicate L operations <LPLR> and <LRLP> transpose Q′ = C minor to $T_l(Q) = T_1(Q)$ = C♯ minor;

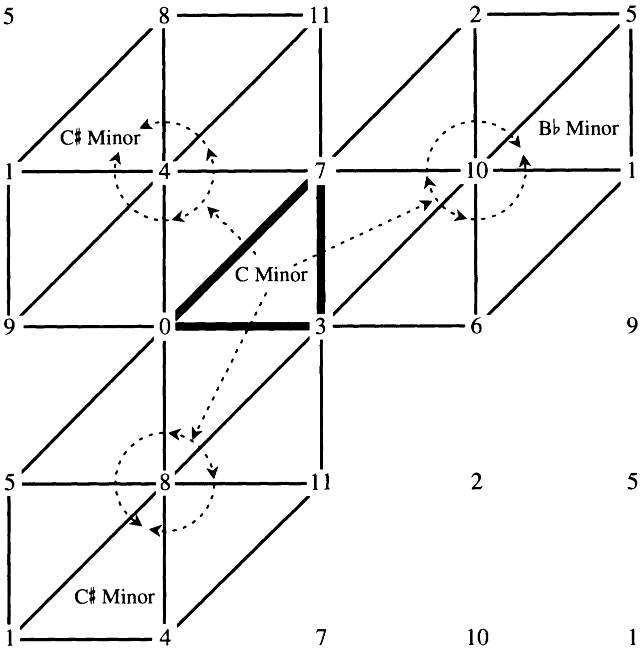


Figure 24b: Quaternary Generators on the Oettingen / Riemann *Tonnetz*

- •To the northwest of Q′, duplicate P operations <PLPR> and <PRPL> transpose Q′ = C minor to T_ρ (Q) = $T_1(Q)$ = C♯ minor;
- To the east of Q′, duplicate R operations <RLRP> and <RPRL> transpose Q′ = C minor to T_z(Q′) = $T_{-2}(Q)$ = B♭ minor.

Figure 25 (p. 50) presents the six operations of Figure 24b in a format that facilitates examination of individual voices. Rotationally related pairs occupy the same row. Pairs of generators are connected by arrows which indicate a particularly intimate voice-leading relationship: paired generators have identical C-voices, and each G-voice is T_4-related to the E♭ voice of its partner. One surprising aspect of this pair-wise partition of the generators is that it is not identical to the pair-wise partition on the basis of rotational equivalence (i.e., shared duplicate). What principles underlie these observations is still an open question, as are the potential compositional and analytic applications. Readers may enjoy exploring

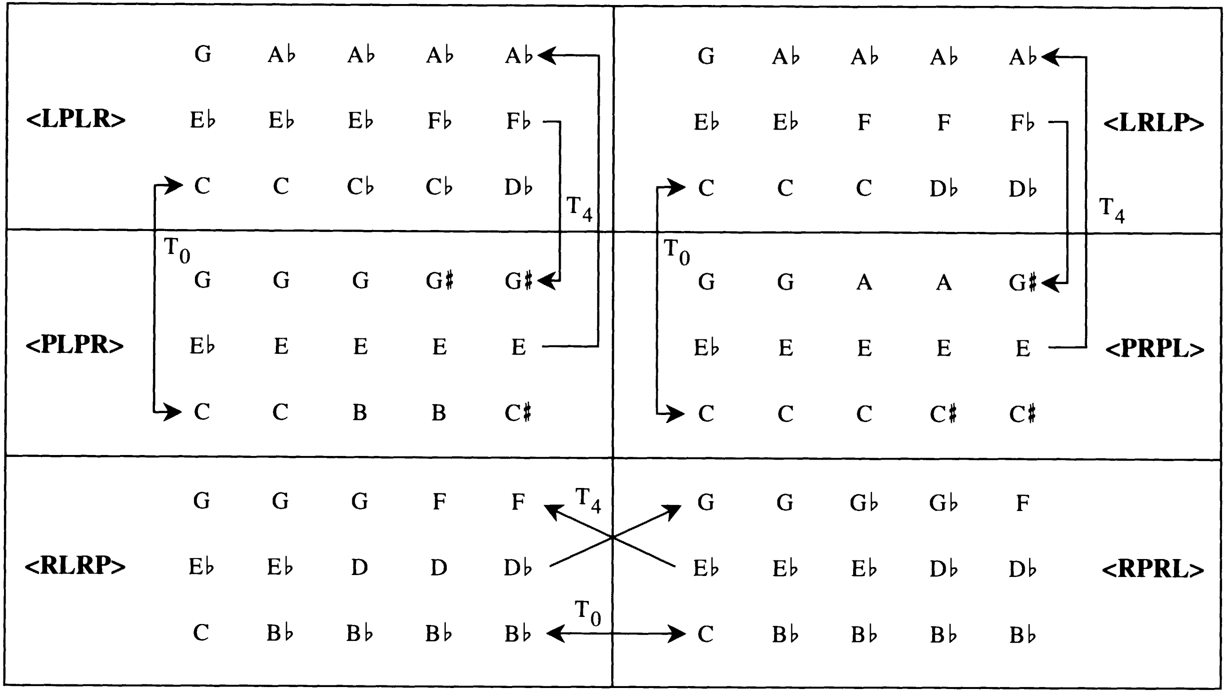


Figure 25: Voice-leading Affinities among Quaternary Generators

the aural kinship between these paired progressions, and between the chromatic sequences that they generate.[28]

Having explored the transpositional behavior of quaternary operations singly iterated, we now study the progressions generated by their recursive application. Complete quaternary-generated cycles of triads are so lengthy as to be of negligible musical value and are impractical to represent in a compact graphic space. Consequently, I will focus here on cycle-segments, or chains, which present one way to conceptualize the stepwise chromatic sequences favored by nineteenth-century composers. Figure 26 (p. 52) superimposes one chain of each rotation-class onto the generic *Tonnetz*.

- <PLPR>, which represents a T_{p} sequence, proceeds from 5:00 to 11:00, moving along the main diagonal in the same direction as its hypotenuse-traversing duplicate, P;
- <LPLR>, which presents a T_{ℓ} sequence, proceeds from 1:00 to 7:00, approximately in the same vertical direction as its duplicate, L;
- <RLRP>, which presents a T_{r} sequence, proceeds from 8:00 to 2:00, approximately in the same horizontal direction as its duplicate, R.

It is characteristic of actual compositional settings that extended sequences are heard not in terms of the continuous flow implied by Figure 26, but rather as a series of terraces. Each terrace is individually unified, but the junction between them characteristically moves "across the barline," inaugurating a new iteration of the sequenced segment at a metrically marked position. Potentially any of the four operations in a quaternary generator can assume the transitional (or external) role, leaving the remaining three operations to assume the role of coherently articulating each terraced region.

A significant feature of quaternary generators is that these terraced regions tend to be coherent in terms of the binary and ternary chains discussed in Sections 3.4 and 3.6. Figure 27 (pp. 54–57) illustrates this point, transferring a representative chain from Figure 26, <PLPR>, onto the surface of the Oettingen/Riemann *Tonnetz* in four different ways. The four representations have identical content, moving through the same series of triads, but their graphic inflections suggest four different ways of partitioning the chain into terraces. Each *Tonnetz* portrait is accompanied by a schematic notational realization in order to reinforce some of the observed qualities.

- Partition (a) (p. 54): <...(PLP) R (PLP) R (PLP)...>. Each set of four triads joined by <PLP> constitutes a terrace unified by the <LP>-cycle or hexatonic constituency of its triads. R plays the role of effecting the "modulation" between neighboring hexatonic regions.

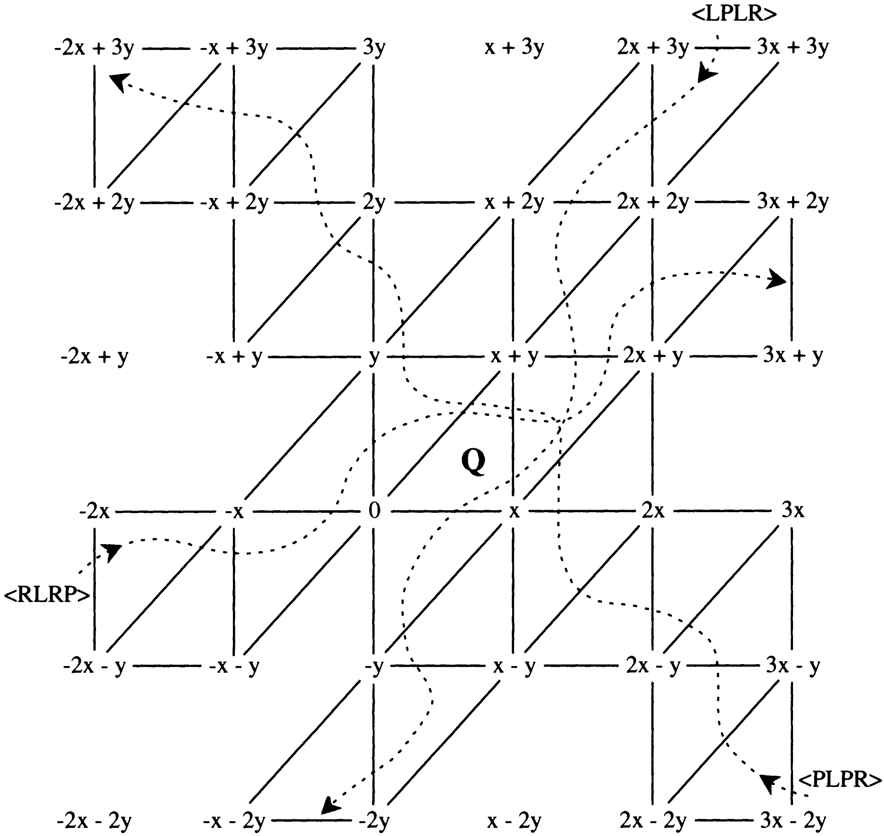


Figure 26: Quaternary Chains on the Abstract *Tonnetz*

The structure is graphically portrayed as a series of unified upward (y-axis) motions, interrupted by a transitional left-ward jog.

- Partition (b) (p. 55): <. . . P (LPR) P (LPR) P (LP . . .>. Each set of four triads joined by <LPR> constitutes a terrace abstractly unified by the LPR-loop constituency of its triads. In concrete musical terms, this unity is insured by the invariant pitch-class inherent to LPR-loops. The P operation external to <LPR> plays the role of "modulating" between adjacent LPR-loops. The modulation is marked by the displacement of the invariant pitch-class that defines the region. (The voice bearing this pc is highlighted in the notational realization by registral placement.) The structure is graphically portrayed as a series of arcs veering northeast of the invariant pc, interrupted by a diagonal jog.

- Partition (c) (p. 56): <...P) L (PRP) L (PRP) L (P...>. Sets of four triads joined by <PRP> constitute terraces unified by the <PR>-cycle or octatonic constituency of the triads. L plays the role of "modulating" between adjacent octatonic regions. The structure is graphically portrayed as a series of leftward (x-axis) motions, interrupted by a transitional upward jog.
- Partition (d) (p. 57): <...PL) P (RPL) P (RPL) P...> An alternative set of LPR-loop segments, with the pitch-classes unifying the regions residing in a different voice than at partition (b). In this case, the loop is articulated via <RPL>. The *Tonnetz* portrays each region as an arc veering southwest of the invariant pc, which again is highlighted in the soprano voice of the notational realization.

The general design of Figure 27 holds for the other quaternary generators, but with a single degree of attenuation: one of the four partitions articulates a series of segments of the <LR> chain. Since the 24 triads are united into a single <LR> chain, the terraces do not define harmonic regions in any obvious way. The design of Figure 27 also transfers to other parsimonious trichords in microtonal systems. For example, the quaternary generator <LPLR> mod-24 leads to a coherent set of terraces for all four partitions, two of which articulate LPR-loop regions unified by pitch-class invariance, the other two of which are articulated by hexatonic-analogue <LP> cycles articulated by <LPL>, and octatonic-analogue <LR> cycles articulated by <LRL>.

This concludes our exploration of generated PLR-family chains in the abstract, and as they apply to consonant triads in the familiar case. Although there are a limited number of independent generators for even cardinalities larger than four, my preliminary investigation suggests that their exploration yields diminishing returns both theoretically and analytically.

4. Some Open Questions

Although our tour of compound PLR-family operations has been densely packed, it has hardly exhausted the terrain. I conclude by suggesting several questions and topics that may reward further investigation. My hope is that some of the investigators might be "microtonal" composers attracted to set-class consistency and smooth voice-leading, and analysts seeking to interpret the triadically based repertory of late

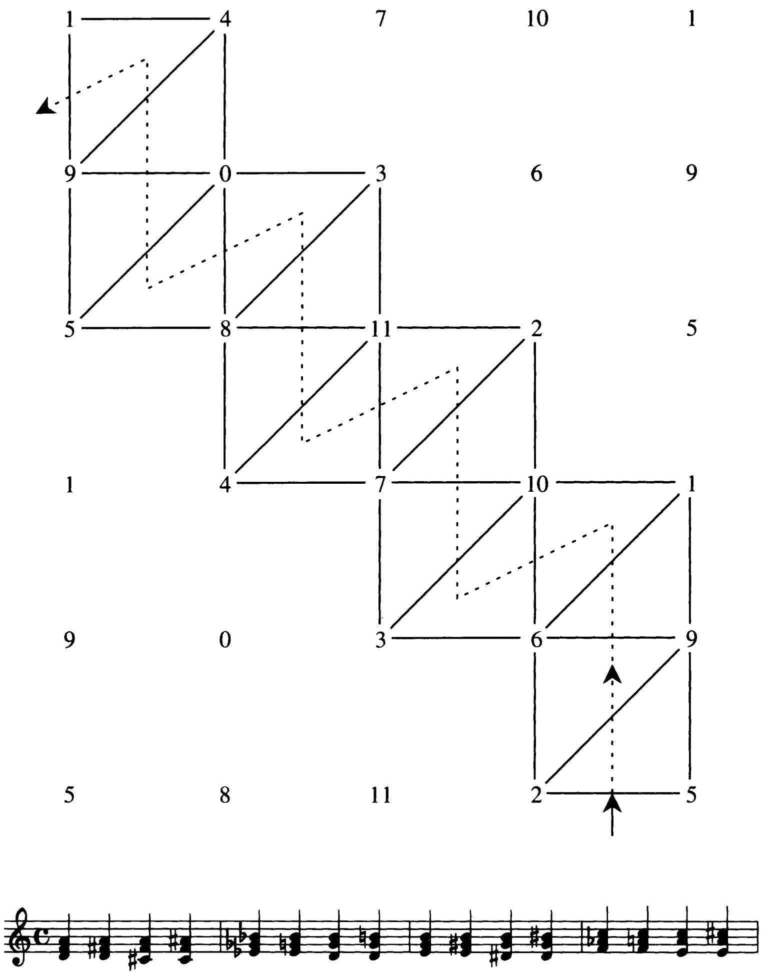


Figure 27 (a)–(d): Four Portraits of a <PLPR> Chain
on the Oettingen/Riemann *Tonnetz*.

Figure 27 (a)

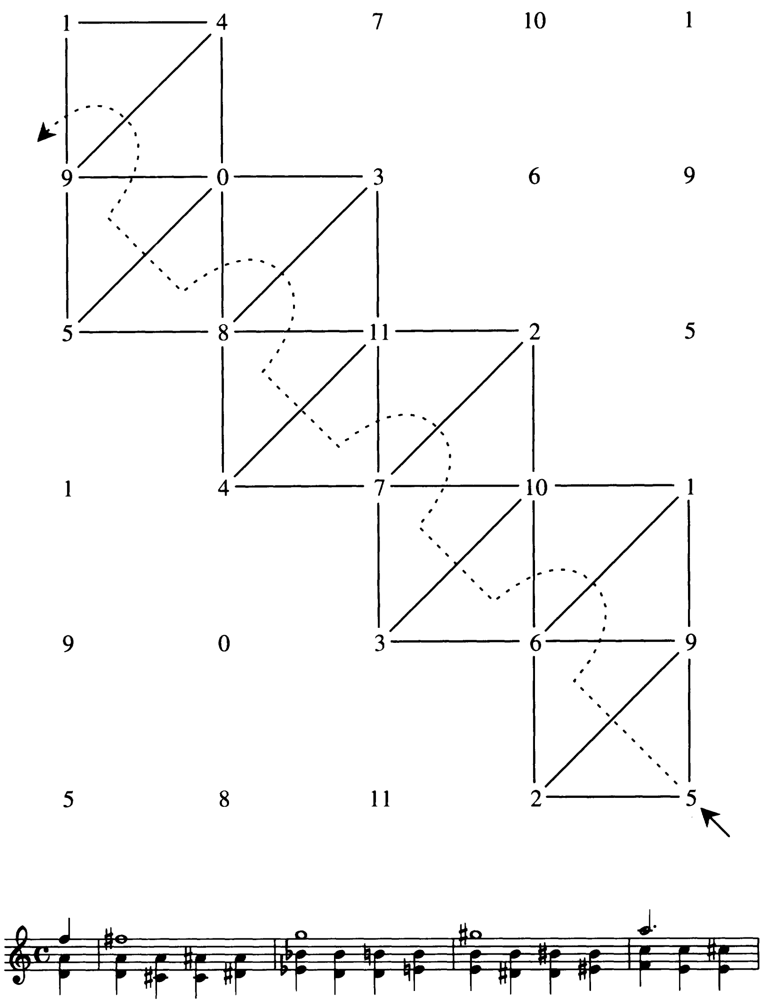


Figure 27 (b)

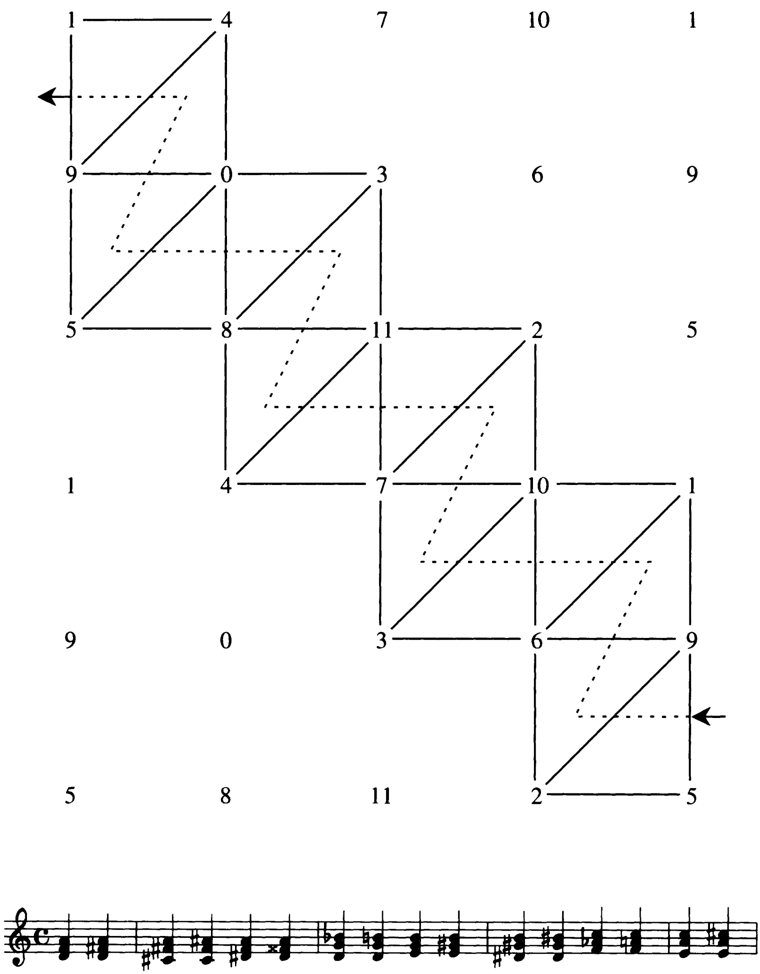


Figure 27 (c)

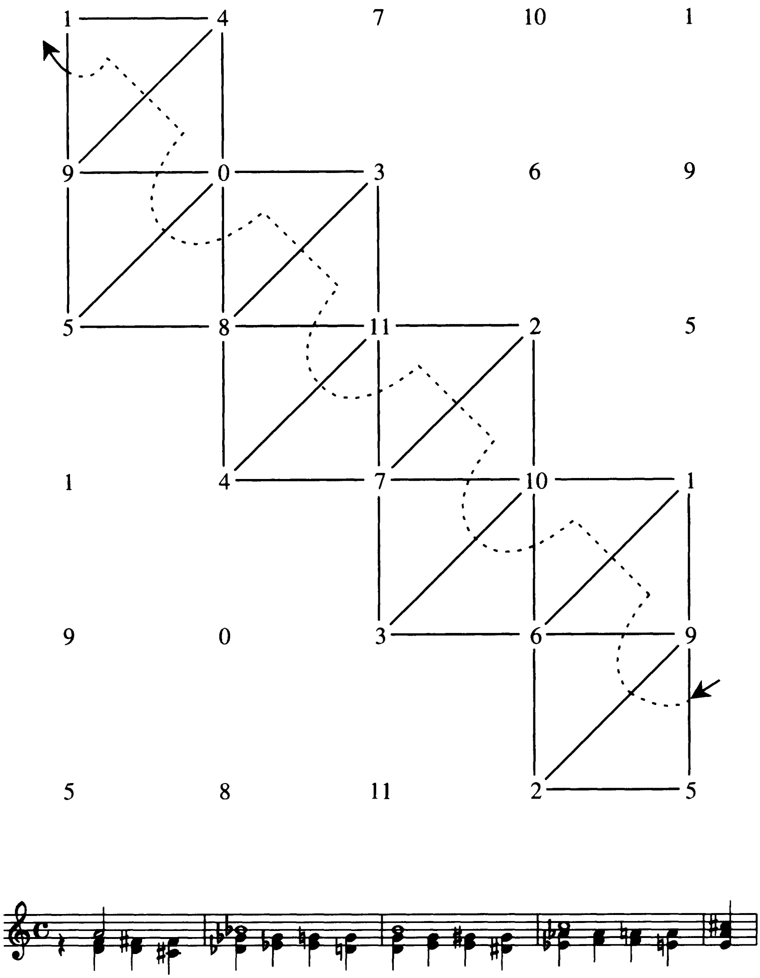


Figure 27 (d)

Romanticism (and perhaps also the fledgling microtonal repertory of our own century). The following comments start down some theoretical and historical paths, but pass by points of analytic interest as well.

(1) Like the work presented here, Balzano (1980) adopts a group-theoretic approach to pitch-relations normally discussed in an acoustic framework, uses a version of the *Tonnetz* to represent these relations, and generalizes properties of c = 12 to other equal-tempered systems. The triadic features that Balzano generalizes differ from those presented here—his generalized triads have step-intervals $<x, x + 1, x^2 - x - 1>$ in a chromatic system of size $x^2 + x$—so that his generalized triads are not the parsimonious ones that have focused our attention here, nor do they inhabit the same set of chromatic systems. It seems worthwhile to explore the relationship between Balzano's generalization and mine, and what is gained and lost by each in relation to the other.

(2) An alternative definition of "parsimony" might open up the category of parsimonious trichords. On the account adopted above, a trichord is parsimonious if its voice-leading is non-zero but minimal under L, P, and R. One alternative avenue that seems promising is to classify a trichord as parsimonious if its voice-leading is non-zero but minimal under **two of the three** PLR-family operations. This would allow not only $\mathscr{p} = 1, \mathscr{l} = 1, \mathscr{r} = -2$ with step-intervals <x, x + 1, x + 2>, but also $\mathscr{p} = 0, \mathscr{l} = 1, \mathscr{r} = -1$ with step-intervals <x, x, x + 1>, and $\mathscr{p} = 1, \mathscr{l} = 0, \mathscr{r} = -1$ with step-intervals <x, x + 1 , x + 1>. On this account, all chromatic systems possess one parsimonious trichord-class.[29] Unlike the parsimonious trichord-classes considered in this paper, the newly yielded trichords are inversionally symmetric, and maximally even in the sense of Clough & Douthett 1991. The new group includes, among others, the set of diatonic (024) triads in a mod-7 diatonic system, and the set of consonant (025) trichords in a mod-8 octatonic system. Furthermore, invoking a distinction made by Lewin (1996), the former group are of the antithesis type, whereas this new group is of the generator type. One advantage of this expanded interpretation of parsimony is that it suggests a way to collapse Lewin's distinction, at least in the case of trichords.

(3) The crooks in the arrows of Figure 11 and related figures suggest that compound operations proceed in multiple stages through a set of intermediate terms, rather than directly to their target triad. I included the crooks to facilitate the tracing of compound operations, but pedagogy should not be confused with ontology. The ontological problem is most generally framed in terms of a path/goal duality . Given some triadic progression <C+, A♭+>, whose most economical PLR-family analysis is <PL>, is the progression to be intuited as a single motion $C+ \xrightarrow{<PL>} A\flat+$, or as a pair of Gestalts $C+ \xrightarrow{P} C- \xrightarrow{L} A\flat+$ whose median term is elided out? What relation holds between the two interpretations, and what is the status of C-?

Along the same lines, how can we choose between two equally economical analyses of a single progression, for example, $f \xrightarrow{<LPR>} E+$ and $f \xrightarrow{<RPL>} E+$? Furthermore, does economy always override other "preference rules" in the assignment of a PLR analysis to a triadic progression? Again to ground the question by example: is <PL> always and only the appropriate analysis for <C+, Ab+>, or are there circumstances under which some other descriptively accurate analysis would be considered more appropriate, for example, $<LP>^2$, <RRPL>, <PRRL>, <PLRR>, or $<RL>^4$? The problem is a general one inherent to the interpretation of relations on a two-dimensional matrix such as a map or a chessboard, and it shows up in nineteenth-century *Tonnetz* analyses (cf. note 24 above) and in other neo-Riemannian transformational accounts of triadic motion (e.g. Lewin 1992, Hyer 1995, Mooney 1996).

(4) The final topic is a historical one. It is apparent that a group-theoretic orientation toward musical materials and their relations did not arrive fully formed in the twentieth century, but rather emerged from elements that had been long present, if not fully articulated or mobilized. What role do the over-determined triad and the over-determined *Tonnetz* play in this emergence?

The responsiveness of the *Tonnetz*, designed to model acoustic relations, to a group-theoretic model potentially furnishes a lever for prying apart the acoustic from the group-theoretic aspects of triadic progressions, and for exploring the cohabitation of a nascent and tacit group-theoretic perspective with an explicitly acoustic one in nineteenth-century harmonic theory. To what extent did nineteenth-century theorists unwittingly smuggle an implicitly group-theoretic orientation beneath the cover of an apparatus whose essentially acoustic nature they never doubted for a moment? Clues can be found not only in the writings of *Tonnetz* navigators, but also in the way that common-tone preservation and incremental voice-leading is treated by nineteenth-century theorists such as Reicha, Fétis, Marx, Hauptmann, Weitzmann, Helmholtz, Tchaikovsky, and Hostinský. This investigation might give new meaning to the time-worn adage that "the seeds of the tonal system's destruction were sewn from within." What is suggested is that the sower is none other than the structure most emblematic of that system, the triad itself. The association of the triad with divinity and perfection as initially conceived by Lippius, and carried down through Schenker and beyond, suggests an allegorical interpretation which is best carried out by scholars trained in theology and the history of ideas.

APPENDIX

The proofs for Theorems 2 and 3 use the formulas for composition of transposition and inversion provided in Rahn (1980, 52), translated into an ordered-set format as follows:

(F1) $<T_a,T_b> = T_{a+b}$;
(F2) $<T_aI,T_b> = T_{a+b}I$;
(F3) $<T_a,T_bI> = T_{b-a}I$;
(F4) $<T_aI, T_bI> = T_{b-a}$

To use these formulas, we will need to translate Lewin's convention for labeling inversions (see Def. (3)) into the index number convention used by Rahn. The translation proceeds as follows:

(F5) $I_v^u = T_{u+v}I$;
(F6) $T_nI = I_n^0$.

* * *

THEOREM 2. Proof that, if #H is odd, $H^2 = T_0$.

(1) $H = <H_1,H_2,\ldots H_n>$, where n is odd.

H_1 is an inversion, by virtue of being a PLR-family operation. The cardinality of $<H_2,\ldots H_n>$ is even, and hence $<H_2,\ldots H_n>$ is a transposition (cf. Section 3.1.2). Thus

(2) $H = <I_b^a,T_p>$.

A second iteration of H likewise can be analyzed as an inversion followed by a transposition, and so

(3) $H^2 = <I_b^a,T_p,I_d^c,T_q>$.

Although the inversional operations are identical in PLR terms, the intervening transposition causes the second inversion to invert around an axis T_p-related to that of the first inversion. Hence

(4) c = a + p; d = b + p.

The transpositional operations are likewise identical in PLR terms, and thus necessarily transpose by the same magnitude, but not necessarily in the same direction. Since one inversion operation intervenes between T_p and T_q, the trichord subject to T_q is inversionally related to the trichord subject to T_p, and so q = -p. Thus:

(5) $H^2 = \langle I_b^a, T_p, I_{b+p}^{a+p}, T_{-p} \rangle$.

(6) $I_b^a = T_{a+b}I$	By translation via F5
(7) $\langle T_{a+b}I, T_p \rangle = T_{a+b+p}I$	F2
(8) $H^2 = \langle T_{a+b+p}I, I_{b+p}^{a+p}, T_{-p} \rangle$	By substitution to (5)
(9) $I_{b+p}^{a+p} = T_{2p+a+b}I$	By translation via F5
(10) $\langle T_{a+b+p}I, T_{2p+a+b}I \rangle = T_{(2p+a+b)-(a+b+p)} = T_p$	F4
(11) $H^2 = \langle T_p, T_{-p} \rangle$	By substitution to (8)
(12) $H^2 = T_0$	F1

QED

* * *

THEOREM 3. Proof that, if ordered set H = T_p, and Ret(H) is the retrograde of H, then Ret(H) = T_{-p}.

(1) If H = T_p, then H can be partitioned into $\frac{\#H}{2}$ transpositions (cf. 3.1.2).

(2) And so $H = \langle T_a, T_b, \ldots, T_m, T_n \rangle$, where $a + b + \ldots m + n = p$.

(3) Each transposition T_q of H is comprised of two inversion operations; hence $T_q = \langle T_{q'}I, T_{q''}I \rangle$.

(4) Thus $H = \langle\langle T_{a'}I, T_{a''}I \rangle, \langle T_{b'}I, T_{b''}I \rangle, \ldots, \langle T_{m'}I, T_{m''}I \rangle, \langle T_{n'}I, T_{n''}I \rangle\rangle$

(5) It follows from (3) that $q = q'' - q'$, via F4.

(6) And so H = $\langle T_{a''-a'}, T_{b''-b'}, \ldots T_{m''-m'}, T_{n''-n'} \rangle$.

(7) Since (4) analyzes H to its atomic elements, it follows that Ret(H) = $\langle\langle T_{n''}I, T_nI \rangle, \langle T_{m''}I, T_{m'}I \rangle, \ldots, \langle T_{b''}I, T_{b'}I \rangle, \langle T_{a''}I, T_{n'}I \rangle\rangle$

(8) Via F4, Ret(H) = $\langle T_{n'-n''}, T_{m'-m''}, \ldots T_{b'-b''}, T_{a'-a''} \rangle$

(9) It follows from (5) that $q' - q'' = -q$.

(10) Thus Ret(H) = $\langle T_{-n}, T_{-m}, \ldots T_{-b}, T_{-a} \rangle$

(11) It follows from (2) that $-n - m \ldots -b - a = -p$.

(12) Then Ret(H) = T_{-p}, via F1

QED

NOTES

The work presented here has benefitted from sustained conversations and correspondence with David Clampitt, John Clough, Jack Douthett, David Lewin, and Michael Siciliano. Adrian Childs provided a helpful reading of the final draft.

1. Throughout this presentation, "triad" restrictively refers to the class of harmonies that is more specifically denoted by the terms "consonant triad," "harmonic triad," "*Klang*," or "member of Forte-class 3–11."
2. Lewin 1982, 1987. The first stages in the development of neo-Riemannian theory are traced in Kopp 1995, 254–269.
3. Lewin 1992; Hyer 1989, 1995. The relation of these terms to those used by Riemann can be cause for confusion. Riemann's "Parallelklang" is equivalent to the Relative major or minor in current English usage; what we call a parallel relation, Riemann terms a "Variant."
4. The term "parsimony" is used in this context in Hostinský 1879, 106. In Schoenberg's writings, the same principle is referred to as the "law of the shortest way." See Schoenberg 1983 (1911), 39. For a formulation of this "law" dating from the late seventeenth century, see Masson 1967 (1694), 47.
5. See Dahlhaus 1990 (1968), 73–74, 86–87, 241. Beginning with (at latest) the eighteenth century, the normative status of common-tone retention and stepwise motion is not only statistical but cognitive: one conceives of them as occuring even when the actual leading of the "voices" violates them, e.g. when instantiations of the common or step-related pitch-classes are realized in different registers.
6. There is a single exception: the 3-12 [048] class. In a certain sense, the exercise does not apply to this class, since those members of 3-12 that share common tones are not distinct from each other. In a different sense, we can view {C,E,G♯} as holding two tones—it matters not which two—in common with its inversion around C (or around any other "even" pc), while the third voice "progresses" by the interval of zero semitones. Here the voice-leading is parsimonious indeed. And no money is spent by the dead.
7. Definition (3) is from Lewin (1987, 51). For a more flexible definition of Q, see note 13 below.
8. Other aspects of Figure 4 are intriguing and suggestive, although not pertinent to the current project. Compare the following trios of trichord-classes for their step-interval differences: {012, 027, and 036}; {013, 016, and 025}. Note also the mod-3 congruence of the step-interval differences of each trichord. For six of the trichord-classes, including the five that are TnI-invariant, the three step-interval differences are congruent to 0, modulo 3.
9. But see Clough & Douthett 1991 and Agmon 1991, both of which identify special properties of the triad *as an object in modulo-7 diatonic space*.
10. For a valuable recent history of the *Tonnetz*, see Mooney 1996. See also Vogel 1993 (1975).
11. Euler 1926 (1739), 319, 349. The *Tonnetz* was already implied, although not laid out in Euler's geometrically compact form, in Rameau's *Nouveau Système de musique théorique* of 1726. See Popovic 1992, 119, 127.

    A more remarkable harbinger of Euler's matrix is implied by the pitch-class names used in ancient Chinese music. A set of 65 bells from the Zeng state of the Marquis Yi, dating from 433 B.C. but only unearthed by archaeologists in 1978, reveals a system of twelve pitch-classes per octave. The names and tonal functions of pitches, which are transmitted through inscriptions on each bell, indicate octave equivalence. The twelve pitch-classes are named by prefix-suffix combinations which constitute a 4x3 Cartesian product isomorphic with Euler's matrix. There are four prefixes, *gong* = C, *zhi* = G, *shang* = D, and *yu* = A, which are identical to four of the pentatonic degree syllables still in use in China. The three suffixes are ø = no transposition, *jue* = transpose upward by major third, and *zeng* = transpose upward by two major thirds. The following table, which transposes Euler's matrix (from Figure 5a) upward by perfect fifth, and then places Zeng syllables alongside the corresponding European pitch-class names, clarifies the isomorphism:

C	**gong**	G	**zhi**	D	**shang**	A	**yu**
E	**gongjue**	B	**zhijue**	F♯	**shangjue**	C♯	**yujue**
G♯	**gongzeng**	E♭	**zhizeng**	B♭	**shangzeng**	F	**yuzeng**

For information on the Zeng bells, see Falkenhausen 1993, especially Chapter 8, and *Chēn,* 1994.

12. This claim requires some elaboration in light of the recent dissertations of Kopp (1995) and Mooney (1996). In the context of Riemann's theory of functions, the PLR-family relations (where P = Riemann's *Variant* and R = Riemann's *Parallel*) modify one of the three functions. Kopp argues that Riemann's separately developed system of musical syntax, which conceives of triadic relations in terms of *Schritte* and *Wechsel,* is not subordinated to the functional framework, but rather co-exists with it in a relationship of mutual autonomy. The syntactic system assigns different labels to PLR-family relations and conceives of them more dynamically than does the system of functions. Although Riemann was not much concerned with *Tonnetz* representations of the syntactic operations, Mooney nonetheless argues that the system of *Schritte* and *Wechsel* is deeply intertwined with the geometry of the Table (see 236–268, esp. 266.) Yet Riemann never bestowed a privileged position to PLR-family operations in the context of his syntactic system, partly because he increasingly interpreted the objects in the table as tonics, and so triangular representations ceased to be relevant.
13. "Pitch or pitch-class" because we have not yet committed the matrix to a modular congruence. To make Figure 6 susceptible to a pitch-space interpretation, Def. (2) would need to be made more flexible as follows: "Q is a trichord $\{0, x, x + y\}$ such that $0 < x \leq y$."
14. To my knowledge, the earliest recognition of the toroidal nature of an equally tempered version of the *Tonnetz* is Lubin 1974. I thank Judith Schwartz for recognizing the relevance of this work to my research.
15. John Clough and David Clampitt were the first to enumerate parsimonious trichords on the basis of their step-interval properties. Their insights were communicated to me by Clough in a letter dated June 25, 1993.
16. Weitzmann 1853, 23. An unbounded version of Figure 9a, implicitly projecting into a torus, was presented in Balzano 1980, 72. The x and y axes of Balzano's matrix are swapped in relation to Figure 9a; otherwise the structures are identical.
17. For background, see Morris 1987, 132.

18. This material builds on Hyer's explorations of the algebra and group structure of neo-Riemannian transformations, in "Tonal Intuitions" and "Reimag(in)ing Riemann."
19. This dictum is axiomatic for theorists of personality, culture, and geo-politics. Its venerable roots are suggested by the following ancient allegory: "The bear went over the mountain to see what he could see. The other side of the mountain was all that he could see."
20. Cohn 1996 cites a number of studies of 6-20; to this list should be added the more recent treatment in van den Toorn 1995: 123–142.
21. Although composed for *Die Zauberharfe*, this music was initially published as the "Rosamunde Overture," under which title it is more familiar.
22. Logier 1976 (1827), p. 38. I am grateful to Scott Burnham for recognizing the significance of Logier's treatise to my research.
23. Michael Siciliano presents a compelling argument for the source status of the <LR> cycle in unpublished work which is currently in preparation as a dissertation.
24. See Rothstein 1991 for an indication of the long reach of elisions in music-theoretic writings. Sources particularly germane to invocation of elisions in the current context include Hauptmann 1991 (1853), 57, 160–161; Oettingen 1866, 145–48; and Hostinský 1879, 103–05.
25. Popovic 1992 refers to the set of pitch-classes that share triadic membership with some central pc as its "neighborhood." The triads of an LPR-loop are grouped together in relation to the central pc (although not necessarily arranged cyclically) by theorists who view triads as assemblies of acoustic consonances, rather than as directly generated entities. See Helmholtz 1954 (1862, 1877), 212 and Hostinský 1879, 67.
26. The interlocking LPR loops correspond with results of experiments reported in Krumhansl and Kessler 1982 and Krumhansl 1990, Chapter 2. Krumhansl and Kessler asked subjects to rate the goodness of fit between a given pitch-class and a given key on the basis of acoustic input, and used that data to profile similarity relations among the 24 keys. Although the profile is best portrayed in four dimensions, their two-dimensional "flattening" of that figure—a rectangle implying a torus—is a rotated version of Figure 22.
27. Similar structures are explored for mod-12 trichords in Bennighof 1987 and Lewin 1996.
28. It is of historical interest that the P-duplicate operations in Figure 25 lead to C♯ minor, but the L-duplicates lead to D♭ minor. This enharmonic distinction vestigially reflects an acoustic one. Recall that P-duplicate and L-duplicate operations lead to different *Tonnetz* locations in Figure 24. For many nineteenth-century theorists (and some modern ones as well, e.g. Vogel 1993 (1975)) the notational and locational distinctions would reflect an acoustic distinction of three syntonic commas.
29. These were first enumerated by Clough and Clampitt in the same document referred to in note 15 above.

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